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MADER'S CONJECTURE ON EXTREMELY CRITICAL GRAPHS

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A non-complete graph G is called an (n,k)*-graph* if it is n-connected but $G-X$ is not $(n-|X|+1)$-connected for any $X \subseteq V(G)$ with $|X| \leq k$. Mader conjectured that for $k \geq 3$ the graph $K_{2k+2}-(1-factor)$ is the unique $(2k,k)$-graph (up to isomorphism).

Here we prove this conjecture.

1. Introduction

All graphs considered here are supposed to be finite, undirected, and simple. For terminology not defined here see [3] or [2]. A T*-fragment* of a graph G is the union of the vertex sets of at least one but not of all components of G–T, where T is a *smallest separating set* of G, that is, a separating set consisting of $\kappa(G)$ many vertices, where $\kappa(G)$ denotes the *(vertex-)connectivity* of G. A *fragment* is a T-fragment for some T. Note that for any T-fragment F, $T=N_G(F)$ holds, so T can be reconstructed from F. Let n and k be non-negative integers. A non-complete graph G is called k*-critically* n*-connected* or, briefly, an (n,k)*-graph*, if G is n-connected and for all $X \subseteq V(G)$ with $|X| \leq k$, $G-X$ is not $(n-|X|+1)$-connected. Hence a non-complete graph G is an (n,k)-graph if and only if $\kappa(G-X) = n-|X|$ for all $X \subseteq V(G)$ with $|X| \leq k$, or, alternatively, if and only if $\kappa(G)=n$ and every set of at most k vertices is contained in a smallest separating set of G. This concept has been introduced by Maurer and Slater [18] and has been within the scope of research interest for more than two decades. Their long standing

Mathematics Subject Classification (2000): 05C40, 05C35, 05C75 conjecture that there is no (n,k)-graph for $n<2k$ [18] has been answered affirmatively by Su [19], and the second conjecture in [18], stating that there is no 3-critically n-connected line graph, has been answered affirmatively by the author [6]. However, the sustainable attacks to these two ex-conjectures led to several stronger ones, which are still open in general.

One of these conjectures suggests that the "extremely critical graphs", i. e. the $(2k,k)$-graphs, are uniquely determined for each $k\geq 3$ ([14, Conjecture 1a]). ("Uniqueness" here always means "uniqueness up to isomorphism".) Theorem 42 of this paper gives an affirmative answer.

2. Proof Strategy

Let me briefly outline the proof strategy.

Roughly, we fix k, consider a $(2k,k)$-graph G on v vertices, and exclude all values $v>2k+2$ from the play.

From earlier results we will derive easily that G has a vertex x of degree $2k$. It is known that $G-x$ is a $(2k-1,k-1)$-graph and every fragment of $G-x$ must contain a neighbor of x. So $G-x$ can't possess a system of more than $2k$ disjoint fragments.

The basic argument now is that for v being large enough, a $(2k-1,k-1)$-graph on $v-1$ vertices must have a large system of disjoint fragments of cardinality 1 or 2 (in fact, more than $2k$ disjoint fragments). This can be obtained by a result on the existence of a possibly smaller but still sufficiently large system in a $(2k-2,k-2)$-graph on $v-2$ vertices, which can in turn be derived from the existence of such a system in a $(2k-3,k-3)$-graph on $v-3$ vertices, and so on. The ratio $(2k-j)/(k-j)$ increases as $j<k$ increases, and as there are $(6k,k)$-graphs without fragments of cardinality 1 or 2, the iteration has to stop at some point. We choose the number j of steps in such a way that we still can *guarantee* one or two fragments of cardinality 1 or 2 in a $(2k-j,k-j)$-graph on $v-j$ vertices.

This argument works, in principle, for $k\geq 5$ and $v\geq 2k+6$, where some additional work is needed for several small values of k. The "small orders" $v\in\{2k{+}3,2k{+}4,2k{+}5\}$ are excluded by a different and much easier argument.

The paper is organized as follows. The next section will present preliminary results and further conjectures on (n,k)-graphs. In Section 4 we exclude, for $k\geq 3$, the existence of $(2k,k)$-graphs of order $2k+3$, $2k+4$, and $2k+5$. In Section 5 and Section 6 we prove several theorems on the existence of disjoint small fragments in certain (n,k)-graphs, where n and, sometimes, the order of the graphs in question is bounded from above by certain linear functions of k. These results are the starting points for iterations which produce lower bounds to the number of disjoint small fragments in $(2k-1,k-1)$-graphs of order at least $(2k-1)+6$. The iteration step is the combinatorial heart of the proof and described in Section 7. In the analysis of the iterations in Section 8 it turns out that for $k\geq 5$ an "almost extremely critical graph", i.e. a $(2k–1,k–1)$-graph, of order at least $2k+5$ has more than $2k$ disjoint fragments of cardinality 1 or 2 (which is interesting on its own).

An interesting aspect of the iteration is that most of the contribution to the bound to the number of disjoint fragments in an almost extremely critical graph is obtained by "rounding fractions to the top", which makes the analysis little harder.

I would like to mention already here that the rather particular results of Section 6 are designed to settle the case $k\in\{5,6,7,10\}$. If one is interested in the other case only, one may skip this section.

3. Preliminary Results

Let us start with preliminary results and further conjectures. The first lemma is a "generalization" of Lemma 1(a) and Lemma 1(b) in [15]. Although the proof and the following ones are standard, I would like to include them here. For a graph G and $X\subseteq V(G)$, let $N_G(X) := \{y\in V(G)-X : y$ is adjacent to some $x\in X\}$ be the *neighborhood* of X in G, and let $\overline{X}^G := V(G)-(X\cup N_G(X))$. For $x\in V(G)$, let $N_G(x) := N_G(\{x\})$. In almost all situations, the index G is clear from the context and will be omitted. Note that if F is a T-fragment of some graph then so is $\overline{F}$, its *complement* or *complementary fragment*.

Lemma 1 ([15]). *Let F be a T-fragment of a graph G of connectivity n and let $X\subseteq V(G)$ such that $|N(X)|=n$ and such that $F\cap X\neq\emptyset$.*

Then $|F\cap N(X)|\geq|\overline{X}\cap T|$ and $|X\cap T|\geq|\overline{F}\cap N(X)|$.

If equality holds in one or both of these inequalities then $F\cap X$ is a fragment and $N(F\cap X)=(F\cap N(X))\cup(T\cap N(X))\cup(T\cap X)$. In particular, $F\cap X$ and $\overline{F}\cap\overline{X}$ are both fragments if and only if they are both non-empty.

Proof. Since $\emptyset\neq\overline{F}\subseteq\overline{F\cap X}$, $N(F\cap X)$ is a separating set of G, and since $N(F\cap X)\subseteq(F\cap N(X))\cup(T\cap N(X))\cup(T\cap X) =: L$, we obtain $n\leq|N(F\cap X)|\leq|L|$.

Since $|L| = |T-\overline{X}|+|F\cap N(X)| = n-|\overline{X}\cap T|+|F\cap N(X)|$ it follows $n\leq|N(F\cap X)|\leq|L|=n-|\overline{X}\cap T|+|F\cap N(X)|$, and, therefore, $|F\cap N(X)|\geq|\overline{X}\cap N(X)|$; if equality holds here then it holds in any of the inequalities above, in particular $|L|=|N(F\cap X)|=n$ and $N(F\cap X)=L$, and so $F\cap X$ is an L-fragment.

Since $|L| = |N(X)-\overline{F}|+|X\cap T| = n-|\overline{F}\cap N(X)|+|X\cap T|$ it follows analogously that $|X\cap T|\geq|\overline{F}\cap N(X)|$ and that if equality holds here then $X\cap F$ is an L-fragment, implying the assertion of the lemma. ∎

A fragment of a graph G is called an *end* of G if it does not contain another fragment of G properly (in particular, ends induce connected subgraphs). The next three lemmata provide some information on distinct intersecting ends or small fragments.

Lemma 2. *If B is an end of a graph G and F is a T-fragment such that $T\cap B\neq\emptyset$ and $F\cap B\neq\emptyset$ then $\overline{F}\cap\overline{B}=\emptyset$, $|B\cap T|>|\overline{F}\cap N(B)|\geq 1$, and $|F\cap N(B)|>|\overline{B}\cap T|\geq 1$. In particular, $|F|\geq 3$ and $|B|\geq 3$.*

Proof. Since neither F nor $\overline{F}$ is contained in the end B, there exist $x\in F-B$, $y\in\overline{F}-B$. There exists an x,y-path whose internal vertices are contained in $\overline{B}$, so $|\overline{B}\cap T|\geq 1$. Since $F\cap B$ is not a fragment, we obtain $|B\cap T|>|\overline{F}\cap N(B)|$ and $|F\cap N(B)|>|\overline{B}\cap T|$ from Lemma 1 (applied to B for X). If $\overline{F}\cap\overline{B}$ was non-empty, then these inequalities would not hold by Lemma 1 (applied to $\overline{F},\overline{B}$ for F,X), hence $\overline{F}\cap\overline{B}=\emptyset$. Since any vertex in $\overline{B}\cap T\neq\emptyset$ must have a neighbor in $\overline{F}$, $\overline{F}\cap N(B)\neq\emptyset$ follows. ∎

Lemma 3. *If B and C are distinct intersecting ends of a graph G of connectivity n then $|B|\geq 3$, $|C|\geq 3$, and $|\overline{B}|+|\overline{C}|\leq|V(G)|-n-2$.*

Proof. Since $B\neq C$ and $B\cap C\neq\emptyset$, we obtain $B\cap N(C)\neq\emptyset$ and $C\cap N(B)\neq\emptyset$. From Lemma 2 it follows $\overline{B}\cap\overline{C}=\emptyset$, $|B|>|B\cap N(C)|>|\overline{C}\cap N(B)|\geq 1$, $|B|\geq 3$, and $|C|\geq 3$. By Lemma 2, at least one of $B\cap\overline{C}$, $C\cap\overline{B}$ is empty. Hence either $\overline{B}\subseteq N(C)$ or $\overline{C}\subseteq N(B)$.

Without loss of generality, $\overline{C}\subseteq N(B)$, implying that $|B|\geq|B\cap C|+|B\cap N(C)|>|B\cap N(C)|>|\overline{C}\cap N(B)|=|\overline{C}|$, hence $|B|\geq|\overline{C}|+2$, and, finally, $|V(G)|-n=|\overline{B}|+|B|\geq|\overline{B}|+|\overline{C}|+2$, implying the assertion. ∎

Lemma 4. *The intersection of distinct fragments of cardinality 2 of a graph G is either empty or a fragment of cardinality 1 of G.*

Proof. Let C,F be distinct fragments of cardinality 2 and suppose that $F\cap C\neq\emptyset$. Then $|F\cap C|=1$. We may assume that both C and F contain a vertex of degree $n+1$. In particular, C and F are connected and, thus, $C\cap N(F)\neq\emptyset$. Since the vertex in $C\cap N(F)$ must have a neighbor in $\overline{F}$, which can not be contained in C and, therefore, must be contained in $N(C)$, $1\geq|C\cap N(F)|\geq|\overline{F}\cap N(C)|\geq 1$ follows. Hence $|C\cap N(F)|=|\overline{F}\cap N(C)|$ and, by Lemma 1 (applied to C for X), $F\cap C$ is a fragment, and has cardinality 1. ∎

Let G be an (n,k)-graph and $Z \subseteq V(G)$. If $\kappa(G-Z)=n-|Z|$ then it turns out easily that every fragment of $G-Z$ is a fragment of G as well, whose neighborhood in G has to contain Z. If $|Z|\leq k$ then $G-Z$ is an $(n-|Z|,k-|Z|)$-graph, as one can prove with slightly more effort. However, it is often useful to "reduce criticity" not by $|Z|$ when removing Z from the graph but by something less than $|Z|$. The next two lemmata describe settings where we can find such a set. With a slightly weaker condition to k in Lemma 6, they appeared in [7], and the proofs are literally the same as there.

Lemma 5 **([7, Lemma 3]).** *Let G be an (n,k)-graph and $X,Y \subseteq V(G)$ with $|X|\leq k$ such that $Y \subseteq T$ for every smallest separating set T of G with $X \subseteq T$. Then every fragment of $G-(X\cup Y)$ is a fragment of G, and $G-(X\cup Y)$ is an $(n-|X\cup Y|,k-|X|)$-graph.*

Lemma 6 **([7, Lemma 4]).** *Let G be an (n,k)-graph and let $\{x,y\}$ be a fragment of G.*

1. *If $k\geq 1$ and x and y are not adjacent or x or y has a degree exceeding n then $G-\{x,y\}$ is an $(n-2,k-1)$-graph.*
2. *If $k\geq 2$ and $w\in N(\{x,y\})-N(y)$ then $G-\{w,x,y\}$ is an $(n-3,k-2)$-graph.*
3. *If $k\geq 2$ and $w\in N(\{x,y\})-N(y)$ and $z\in N(\{x,y\})$ is adjacent to all vertices in $N(\{x,y\})-\{w,z\}$ then $G-\{w,x,y,z\}$ is an $(n-4,k-2)$-graph.*

In either case, the fragments of the reduced graph are fragments of G.

The following Conjecture appeared in [14].

Conjecture 7 **([14]).** Every (n,k)-graph has a system of $2k+2$ disjoint fragments.

One can't expect more disjoint fragments in general, as it is shown by the complementary graph of $k+1$ disjoint copies of a complete bipartite graph $K_{p,q}$, $p,q\geq 1$, which is an $(kp+kq,k)$-graph with exactly $2k+2$ fragments.

For $k\leq 2$, Conjecture 7 is true by the results of [15] and [20]. This will be of importance in Section 6, hence we give the statements separately.

Theorem 8 **([15], [20]).** *Every $(n,1)$-graph has four disjoint fragments.*

Theorem 9 **([20]).** *Every $(n,2)$-graph has six disjoint fragments.*

Clearly, Conjecture 7 is equivalent to the conjecture that every (n,k)-graph has a system of $2k+2$ disjoint *ends*. If we weaken there the condition of being *disjoint* to the condition of being *distinct* then there is the following, positive result, which has first been proved by Su. Later, Jordán found a shorter proof. An *end dominating set* or an *end cover* of a graph G is a set of vertices of G which intersects every end of G.

Theorem 10 **([21], [5]).** *Every end cover of an (n,k)-graph has cardinality at least $2k+2$.*

In particular, every (n,k)-graph has $2k+2$ distinct ends.

As it as been showed in [21] and [5], Theorem 10 provides almost immediately an answer to the existence problem for (n,k)-graphs mentioned above:

Theorem 11 **([21], [5]).** *There is no (n,k)-graph for $n<2k$.*

Since it is known that every $(n,2)$-graph is finite [13], the restriction of our considerations to *finite* graphs is not essential; however, there are infinite $(n,1)$-graphs for every $n \geq 1$.

Furthermore, it can be derived from Theorem 10 that Conjecture 7 is true for extremely critical graphs [5]:

Theorem 12 **([5]).** *Every $(2k,k)$-graph has a system of $2k+2$ disjoint ends.*

If Conjecture 7 would be true for some $k=\ell$ then the following is true for $k=\ell+1$, as it has been proved in [14].

Conjecture 13 **([14]).** Every (n,k)-graph has a fragment of cardinality at most $\frac{n}{2k}$.

For $k=1$, this can be improved as follows:

Theorem 14 **([15]).** *Every $(n,1)$-graph has two disjoint fragments of cardinality at most $\frac{n}{2}$ each.*

Even when restricted to the case $n<4k$, Conjecture 13 is unsolved.

Conjecture 15 **([14]).** Every (n,k)-graph with $n < 4k$ has a vertex of degree n.

From the results in [9] we know:

Theorem 16 **([9]).** *Every $(n,2)$-graph with $n<8$ contains two vertices of degree n.*

By Theorem 14 and Theorem 16, every (n,k)-graph with $n < 4k$ and $k \in \{1,2\}$ contains even *two* vertices of degree n, which I should conjecture to be true for *all* k.

The following two results plus estimates on the transversal number of certain hypergraphs from [12], [22], and [11] led to Theorem 19, Theorem 20, and Theorem 21 below; since we have to refine Theorem 20 here, I cite parts of this machinery. A fragment F of some graph G is *proper* if $|F| \leq |\overline{F}|$.

Lemma 17 ([19], [11]). *Distinct proper ends of a graph G are disjoint, and at least half of the ends of G are proper.*

Readers which are not familiar with hypergraph terminology are refered to Berge's book [1] or to [11]. An *edge dominating set* or a *transversal* or an *edge cover* of a multihypergraph G is a set of vertices of G which intersects every edge of G, and a set S of edges of G is *independent* if distinct edges of S do not intersect.

Theorem 18. *Let G be a multihypergraph such that every edge of G has cardinality at least 3 and such that there is an independent edge set of cardinality at least $|E(G)|/2$. Then there exists a transversal of G of cardinality at most $\frac{5}{12} \cdot |V(G)|$.*

The following three results have been derived in [11].

Theorem 19 ([11]). *Every (n,k)-graph with $n < \frac{16}{5}k$ has a vertex of degree n.*

Theorem 20 ([11]). *Every (n,k)-graph with $n < \frac{24}{5}k$ has a fragment of cardinality 1 or 2.*

Theorem 21 ([11]). *Every (n,k)-graph with $n < 6k$ has a fragment of cardinality at most 3.*

We conclude this section with the following four results on $(2k,k)$-graphs. The first one follows straightforward from Theorem 2 in [7], and the second one follows from Theorem 19 or a simple induction argument (cf. [8]). The third one summarizes the main results in [7], and Theorem 25 follows from the characterization of $(4,2)$-graphs in Theorem 7 in [14] or, for example, from the considerations in [17].

Theorem 22 ([7]). *Let G be a $(2k,k)$-graph non-isomorphic to $K_{2k+2} - (1 - factor)$ without fragments of cardinality 2. Then $|V(G)| \geq 2k+6$.*

Theorem 23. *Every $(2k,k)$-graph has a vertex of degree $2k$.*

Theorem 24 ([7]). *For $k \in \{3,4\}$, the graph $K_{2k+2} - (1 - factor)$ is the unique $(2k,k)$-graph.*

Theorem 25 ([14], [17]). *Every $(4,2)$-graph is 4-regular.*

4. Extremely critical graphs of small order

Corollary 1 in [8] states that there is no $(2k,k)$-graph of order $2k+3$ for $k\geq 3$. The aim of this section is to remove graphs of order $2k+4$ and $2k+5$ from the list of candidates for a $(2k,k)$-graph, too. This is necessary in the sense that the iteration method described in the later sections *fails* for these orders. A fragment of a graph G is called *trivial* if it consists of a single vertex only (which then has degree $\kappa(G)$). An end F of G such that $N_G(F)=T$ is called a T*-end*.

Throughout, it will be important to find a large system of disjoint fragments in some highly critical graph G. We start with the following Theorem, which gives a structural characterization of $(2k-3,k-2)$-graphs possessing a certain end cover which fail to satisfy Conjecture 7.

Theorem 26. *Let* $k\geq 3$.

Let G *be an* $(2k-3,k-2)$*-graph and suppose that* S *is an end cover of* G *of cardinality* $2k-1$ *such that any non-trivial end intersects* S *in at least two vertices.*

Then G *has no system of* $2k-2$ *disjoint fragments if and only if there are vertices* s_i *and* T_i*-ends* B_i, $i\in\{0,1,2\}$, *such that* $\{s_i\}=(S\cap B_{i+1}\cap B_{i-1})-B_i$ *and*

$$\begin{aligned}
A &:= B_0\cap B_1\cap B_2,\\
M &:= T_0\cap T_1\cap T_2,\\
P_i &:= T_i\cap B_{i+1}\cap B_{i-1},\\
Q_i &:= \overline{B_i}\cap T_{i+1}\cap T_{i-1}, \text{ and}\\
C_i &:= \overline{B_i}\cap B_{i+1}\cap B_{i-1}
\end{aligned}$$

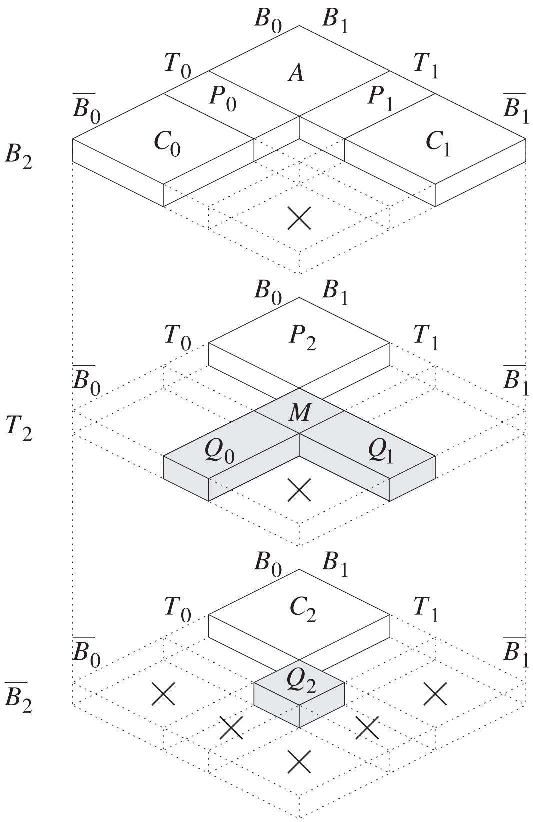
($i\in\{0,1,2\}$*, indices modulo* 3*) form a decomposition of* $V(G)$ *such that* $S-\{s_0,s_1,s_2\}=M\cup Q_0\cup Q_1\cup Q_2$, $|P_i|-1=|Q_i|\geq 1$ *for all* $i\in\{0,1,2\}$, $|\overline{B_i}|=1$ *for at most one* $i\in\{0,1,2\}$, *and if* $C_i\neq\emptyset$ *for some* $i\in\{0,1,2\}$ *then* $C_{i+1}=C_{i-1}=\emptyset$ *and* $|Q_i|\geq|Q_{i+1}|+|Q_{i-1}|+1$.

Proof. If G has a system of $2k-2$ disjoint ends then S contains at least $|S|-2$ trivial ends by the assumptions on S. Then s_i and B_i can not exist, since (disregarding the properties of the induced decomposition of $V(G)$) the B_i would be non-trivial and distinct, and their union would contain at least 3 vertices of S and hence at least one trivial end, which is impossible.

Now suppose that G has no system of $2k-2$ disjoint fragments, let L be the set of vertices of S which form trivial ends in G, and let $N:=S-L$. By Theorem 10, there exists a system of $2k-2$ distinct ends, and we may assume that each trivial end is contained in this system, since a trivial end is disjoint from every end distinct from it; by the same argument, $2k-3$ trivial ends would provide us with $2k-2$ disjoint ends, and hence we conclude $|L| \leq 2k-4$, so $|N| \geq 3$.

Let $N' \subseteq N$ with $|N'| = 2$. By Theorem 10, $S - N'$ is not an end cover. Hence there must be an end B of G such that $B \cap S \subseteq N'$. Since S and, thus, N intersects every non-trivial end of G in at least two vertices it follows $B \cap S = N'$. Taking three arbitrary vertices s_0, s_1, s_2 from N, it follows from these considerations, applied to all three subsets N' of $\{s_0, s_1, s_2\}$ with $|N'| = 2$, that there are T_i-ends B_i for $i \in \{0,1,2\}$ such that $\{s_i\} = (S \cap B_{i+1} \cap B_{i-1}) - B_i$ and $(B_0 \cup B_1 \cup B_2) \cap S = \{s_0, s_1, s_2\}$.

The sets $Z_0 \cap Z_1 \cap Z_2$, where $Z_i \in \{B_i, T_i, \overline{B_i}\}$, clearly form a decomposition of $V(G)$ into 27 parts. This is illustrated in the following picture, and our aim is to show that most of the parts are empty.



Let $i \in \{0,1,2\}$. Since $s_i \in B_{i+1} \cap B_{i-1} \neq \emptyset$, we have $\overline{B_{i-1}} \cap \overline{B_{i+1}} = \emptyset$ by Lemma 2. (This implies that the seven cross-marked parts in the figure above are empty.) Furthermore, the set

$$\begin{aligned} X_i &:= (T_{i+1} - \overline{B_{i-1}}) \cup (T_{i-1} - \overline{B_{i+1}}) \\ &= (T_{i+1} \cap B_{i-1}) \cup (T_{i+1} \cap T_{i-1}) \cup (B_{i+1} \cap T_{i-1}) \end{aligned}$$

must contain at least $\kappa(G)+1 = 2k-2$ vertices, since it contains $N(B_{i+1} \cap B_{i-1})$ and $B_{i+1} \cap B_{i-1}$ is not a fragment. Observe that

$$Y_i := (T_{i+1} - B_{i-1}) \cup (T_{i-1} - B_{i+1})$$

contains $2\kappa(G) - |X_i|$ vertices. Since the vertices of $R := S - \{s_0, s_1, s_2\}$ are not contained in B_{i+1} or B_{i-1} it follows $R \subseteq Y_i$. Now $2k-4 = |R| \leq |Y_i| = 2\kappa(G) - |X_i| = 4k-6-|X_i| \leq (4k-6) - (\kappa(G)+1) = 2k-4$, implying that $R = Y_i$ and $|X_i| = 2k-2$ for all $i \in \{0,1,2\}$. Consequently, $|B_{i+1} \cap T_{i-1}| = |X_i| - (|T_{i+1} \cap T_{i-1}| + |B_{i-1} \cap T_{i+1}|) = |X_i| - (|T_{i+1}| - |\overline{B_{i-1}} \cap T_{i+1}|) = (2k-2) - (2k-3) + |\overline{B_{i-1}} \cap T_{i+1}| = |\overline{B_{i-1}} \cap T_{i+1}| + 1$ for all $i \in \{0,1,2\}$. Since R does not depend on i, every vertex of G is either contained in none of the Y_i or in all of the Y_i. (This implies that the nine unmarked and unlabelled parts in the figure above are empty.) It follows that $R = Y_i = M \cup Q_0 \cup Q_1 \cup Q_2 = S - \{s_0, s_1, s_2\}$ and $|P_i| = |T_i \cap B_{i-1}| = |T_{(i+1)-1} \cap B_{(i+1)+1}| = |\overline{B_{(i+1)-1}} \cap T_{(i+1)+1}| + 1 = |Q_i| + 1$ for all $i \in \{0,1,2\}$.

Hence we may decompose $V(G)$ as it is indicated in the assertion, and it remains to prove that $|\overline{B_i}| = 1$ for at most one $i \in \{0,1,2\}$ and if $C_i \neq \emptyset$ for some $i \in \{0,1,2\}$ then $C_{i+1} = C_{i-1} = \emptyset$ and $|Q_i| \geq |Q_{i+1}| + |Q_{i-1}| + 1$.

Let us derive a contradiction from $|\overline{B_{i+1}}| = |\overline{B_{i-1}}| = 1$ for some $i \in \{0,1,2\}$: from the latter it would follow that $G' := G - (\overline{B_{i+1}} \cup \overline{B_{i-1}} \cup \{s_i\})$ is a $(2k-6, k-3)$-graph, since every smallest separating set of G containing $s_i \in B_{i+1} \cap B_{i-1}$ intersects both $\overline{B_{i+1}}$ and $\overline{B_{i-1}}$. By Theorem 12, G' has a system of $2k-4$ disjoint ends and, taking the two trivial ends $\overline{B_{i+1}}, \overline{B_{i-1}}$ of G into account, G has a system of $2k-2$ disjoint ends, a contradiction.

Finally suppose that $C_i \neq \emptyset$ for some $i \in \{0,1,2\}$. From Lemma 2 we obtain $C_{i+1} = C_{i-1} = \emptyset$. Furthermore, $N(C_i) \subseteq M \cup Q_i \cup P_i$. Since C_i is properly contained in the end B_{i+1}, it follows $|M| + |Q_i| + |P_i| \geq |N(C_i)| \geq \kappa(G) + 1 = (2k-3)+1$. Hence $|Q_i| = |P_i| - 1 \geq 2k-3-|M|-|Q_i| = 2k-3-|R-Q_{i+1}-Q_{i-1}| = |Q_{i+1}| + |Q_{i-1}| + 1$. ∎

In view of Conjecture 7, there should be lots of additional features of the vertex partition in Theorem 26 waiting for their discovery. The result we derive here states that Conjecture 7 restricted to a very particular subclass of $(2k-3, k-2)$-graphs is true. In view of the short proof of Theorem 12 in [5] it is, however, quite tantalizing that I can not do it for $(2k-3, k-2)$-graphs in general.

Theorem 27. *Every $(2k-3,k-2)$-graph G of order at most $2k+3$ which possesses an end cover S of cardinality $2k-1$ such that an end of G is trivial if and only if it intersects S in a single vertex has a system of $2k-2$ disjoint fragments.*

Proof. Let us assume that G as in the statement does not have a system of $2k-2$ disjoint fragments and consider vertices s_0,s_1,s_2, ends B_0,B_1,B_2, and a partition as in Theorem 26. We show that $|V(G)|\geq 2k+4$.

Case 1. $C_i\neq\emptyset$ for some $i\in\{0,1,2\}$. By Theorem 26, $C_{i\pm1}=\emptyset$, $Q_{i\pm1}=\overline{B_{i\pm1}}$, and $|Q_i|\geq|Q_{i-1}|+|Q_{i+1}|+1$. Since at most one of $\overline{B_{i\pm1}}$ has cardinality 1, we obtain $|\overline{B_i}|=|C_i|+|Q_i|\geq 1+(1+2+1)=5$ and $|B_i|=|A|+|P_{i-1}|+|P_{i+1}|\geq 0+2+3=5$, implying that $|V(G)|=|T_i|+|B_i|+|\overline{B_i}|\geq(2k-3)+5+5=2k+7\geq 2k+4$.

Case 2. $C_i=\emptyset$ for all $i\in\{0,1,2\}$. By Theorem 26, $\overline{B_i}=Q_i$ for all $i\in\{0,1,2\}$, and there exists an i such that $|\overline{B_{i\pm1}}|\geq 2$. Hence $|B_i|=|A|+|P_{i-1}|+|P_{i+1}|\geq 0+3+3=6$, implying that $|V(G)|=|T_i|+|B_i|+|\overline{B_i}|\geq(2k-3)+6+1=2k+4$. ∎

The proof of the following consequence of Theorem 27 is more or less the proof of Theorem 1 in [7].

Theorem 28. *For $k\geq 3$, $K_{2k+2}-(1-factor)$ is the unique $(2k,k)$-graph of order at most $2k+5$.*

Proof. We prove this by induction on k. For $k\in\{3,4\}$ it follows from Theorem 24. For the induction step, let $k\geq 5$. We reproduce the proof of Theorem 1 in [7].

Let G be a $(2k,k)$-graph non-isomorphic to $K_{2k+2}-(1-factor)$ and with $|V(G)|\leq 2k+5$. By Theorem 22, G must have a T-fragment $F=\{x,y\}$, $x\neq y$. If one of x,y had a degree larger than $2k$ or x,y were not adjacent then $G-F$ would be a $(2k-2,k-1)$-graph of order at most $2\cdot(k-1)+5$ by 1. of Lemma 6, which is absurd by induction hypothesis. Thus we may asumme that x,y are adjacent and both have degree $2k$. Let w be the vertex in $T-N(y)$ and z be the vertex in $T-N(y)$. Then $G^-:=G-\{w,x,y\}$ is a $(2k-3,k-2)$-graph of order at most $2k+2$ by 2. of Lemma 6, and $T-\{w\}$ is an end cover of G^- of cardinality $2k-1$ such that an end of G^- is trivial if and only if it intersects $T-\{w\}$ in a single vertex (the latter is clear if the end in question does not intersect $\overline{F}$, and follows from Lemma 1 if it does). By Theorem 27, G^- has a system of $2k-2$ disjoint ends $F_1,\ldots,F_{2k-2}$. Each of them has to intersect $N(x)-\{w,x,y\}$ by Lemma 1; $z\in F_i$ implies $|T\cap F_i|\geq 2$. Therefore, at most one of the ends has cardinality at least 2, and any such end must contain z. In particular, w has at least $2k-3$ neighbors of degree $2k$ in $T-\{w,z\}$, and, by symmetry, z has at least $2k-3$ neighbors of degree $2k$ in $T-\{w,z\}$, too.

Let us assume for a while that $|F_i\cap T|\geq 2$ for some $i\in\{1,\ldots,2k-2\}$. Then $F_i\cap T=\{t,z\}$ for some $t\in T-\{w,z\}$. Since $|N(z)\cap(T-\{w,z\})|\geq 2k-3$, $|\overline{F_i}\cap T|=|\overline{F_i}\cap T-\{w,z\}|=1$ and, therefore, $\overline{F_i}=\{t'\}$ for some $t'\in T-\{w,z\}$. Let $T'\supseteq\{w,x,t\}$ be a smallest separating set of G. This implies $y\in T'$. Since F_i is an end in G^- intersected by the smallest separating set $T'-\{w,x,y\}$ of G^-, $\overline{F_i}\cap T'\neq\emptyset$ follows, hence $t'\in T'$. Since z is adjacent to all vertices in $T-\{w,z,t'\}$, z must be in T'. Hence we showed that $T'\supseteq\{w,x,t\}=:X$ implies $T\supseteq\{w,x,y,z,t,t'\}=:Y$. By Lemma 5, $G^{--}:=G-\{w,x,y,z,t,t'\}$ is a $(2k-6,k-3)$-graph of order at most $2\cdot(k-3)+5$. If $k\geq 6$ then this violates our induction hypothesis; if $k=5$ then G^{--} is a $(4,2)$-graph and every vertex of $\overline{F}$ has degree at least 10 in G^{--}, contradicting Theorem 25. Hence the case $|F_i\cap T|\geq 2$ can not occur, implying that all the F_i are trivial ends.

Hence w is adjacent to every vertex in $T-\{w,z\}$, and so is z. It follows that every smallest separating set of G which contains $\{x,y\}$ must contain w,z, too. By 3. of Lemma 6, $G-\{w,x,y,z\}$ is a $(2k-4,k-2)$-graph of order at most $2\cdot(k-2)+5$, violating our induction hypothesis. ∎

5. Existence of small fragments in highly critical graphs

In this section and in the following one, we prove a number of results on the existence of disjoint small fragments in certain (n,k)-graphs. As it has been stated in the introduction, they produce the starting points for the iterations in Section 8.

In [21] it has been proved that Conjecture 13 is true for (n,k)-graphs whose order is sufficiently large in terms of n and k. On the other hand, it is clear that an (n,k)-graph of order at most $n+\frac{n}{k}$ has a fragment of cardinality at most $\frac{n}{2k}$. (In fact, any proper fragment has cardinality at most $\frac{n}{2k}$ then, and so there are at least $k+1$ fragments of cardinality at most $\frac{n}{2k}$ in such a graph (cf. [11]).) The first theorem of this section improves this observation slightly. A fragment A of a graph G is called an *atom* of G if G has no fragments of cardinality less than $|A|$ (so every atom is an end, and if G has trivial fragments then these are the atoms of G).

Theorem 29. *Every (n,k)-graph of order at most $n+\frac{n}{k}+2$ has a fragment of cardinality at most $\frac{n}{2k}$.*

Proof. Let G be an (n,k)-graph of order at most $n+\frac{n}{k}+2$, and let a be the cardinality of an atom of G. Let's assume that $a>\frac{n}{2k}$, hence $a=\frac{n+r}{2k}$ for some $r\geq 1$. By Theorem 11, $a>1$.

We may assume that distinct ends of G are disjoint, for otherwise the cardinalities of their complements would sum up to at most $\frac{n}{k}$ by Lemma 3, which contradicts our assumption. Now it follows from Theorem 10 that G has $2k+2$ *disjoint* ends $A_1,\ldots,A_{2k+2}$. If not all of the A_i were atoms then $|V(G)| \geq (2k+2)\cdot a+1 \geq (2k+2)\frac{n+1}{2k}+1 = n+1+\frac{n+1}{k}+1 > n+\frac{n}{k}+2$, a contradiction. Hence every A_i is an atom. Let T be a smallest separating set of G. If T would intersect at least $2k$ of the atoms A_i, then these would be contained in T and $|T|\geq 2k{\cdot}a \geq 2k{\cdot}\frac{n+1}{2k} = n+1$, which is absurd. Thus, at least 3 atoms do not intersect T, and we find $|V(G)|\geq n+3{\cdot}a$. On the other hand, $|V(G)|\leq n+\frac{n}{k}+2$, which implies that $3a\leq\frac{n}{k}+2$. The latter is equivalent to $n\leq 4k-3r$, which is absurd, since from $a\geq 2$ we obtain $n\geq 4k-r$. ∎

The next lemma states that the system of fragments of cardinality 1 or 2 of a graph G possesses the Helly-property. Later, in Theorem 41, we will generalize it.

Lemma 30. *Suppose that G is a graph without two disjoint fragments of cardinality 1 or 2. Then there exists a vertex which is contained in every fragment of G of cardinality 1 or 2.*

Proof. By Lemma 4, the intersection of distinct fragments of cardinality 2 of G is either empty or a fragment of cardinality 1, so the assertion follows. ∎

We continue with the following Lemma, which will be useful in the investigations of the following section, too.

Lemma 31. *Let B be an end of an (n,k)-graph G with $|B|\in\{3,4\}$. Then one of the following is true.*

1. *$G-B$ is an $(n-|B|,k-1)$-graph, or*
2. *every vertex in B has a neighbor of degree n in $N(B)$ which is adjacent to at least two but not to all vertices of B (in particular, $N(B)$ contains two vertices of degree n), or*
3. *$|B|=4$ and there exists a fragment of cardinality 2 contained in $N(B)$ whose neighborhood contains exactly three vertices of B.*

Proof. Suppose that $G-B$ is not an $(n-|B|,k-1)$-graph. Then there exists an $X\subseteq V(G)-B$ with $|X|\leq k-1$ such that for every smallest separating set $T\supseteq X$ of G which intersects B there exists a T-fragment intersecting B. Suppose that, in addition, 3. does not hold. We have to prove that 2. is true.

Consider a vertex $b\in B$. Since G is an (n,k)-graph, there exists a smallest separating set $T\supseteq X\cup\{b\}$ of G, and there exists a T-fragment F such that $F\cap B\neq\emptyset$.

It follows that $\overline{F}\cap\overline{B}=\emptyset$ and $|B|-1\geq|B\cap T|>|\overline{F}\cap N(B)|\geq 1$ by Lemma 2.

If $\overline{F} \cap B \neq \emptyset$ we had $F \cap \overline{B} = \emptyset$ and $|\overline{F} \cap N(B)| > |\overline{B} \cap T|$ by Lemma 2, implying that $|\overline{B}| = |\overline{B} \cap T| \leq |\overline{F} \cap N(B)| - 1 \leq |B \cap T| - 2 \leq |B| - 4 \leq 0$, which is absurd.

Hence $\overline{F} \cap B = \emptyset$ and, thus, $\overline{F} \subseteq N(B)$, so $1 \leq |\overline{F}| = |\overline{F} \cap N(B)| \leq |B \cap T| - 1 \leq |B| - 2 \leq 2$.

If $|\overline{F}| = 2$ then $|B| = 4$ and $|B \cap T| = 3$ – contradicting our assumption that 3. does not hold.

Hence $|\overline{F}| = 1$, and, therefore, $\overline{F}$ consists of a single vertex t of degree n, which is adjacent to $|B \cap T| \geq 2$ vertices of B but not to all of B, since $F \cap B \neq \emptyset$. Since $b \in T$, t is a neighbor of b, and hence 2. is proved. ∎

Theorem 32. *Every (n,k)-graph of order at most $n+7$ and with $n < 6k$ has two disjoint fragments of cardinality 1 or 2 each.*

Proof. Let G be an (n,k)-graph of oder at most $n+7$ and with $n < 6k$, and let us assume, to the contrary, that G does not have two disjoint fragments of cardinality 1 or 2 each. By Lemma 30, there exists a vertex s such that s is contained in every fragment of cardinality 1 or 2.

Claim 1. If A is a fragment of cardinality 2 and B is an end of cardinality 3 or 4 such that $N(A) \cap B \neq \emptyset$ then $|N(A) \cap B| \geq 3$.

First note that $A \cap B = \emptyset$, for otherwise Lemma 2 would imply $|A| > |A \cap N(B)| > |\overline{B} \cap N(A)| \geq 1$, which is absurd. If $B \cap \overline{A} = \emptyset$ then $B \subseteq N(A)$, thus $|N(A) \cap B| = |B| \geq 3$, and otherwise $\overline{B} \cap A = \emptyset$ and $|B \cap N(A)| > |A \cap N(B)| = |A| = 2$ by Lemma 2. This proves Claim 1.

Claim 2. A does not have a fragment of cardinality 2.

For if A was a fragment of G of cardinality 2 then, by assumption, A does not consist of two vertices of degree n in G. Hence $G - A$ is an $(n-2, k-1)$-graph by 1. of Lemma 6. By Theorem 10, $G - A$ had $2k$ distinct ends. Since they are fragments of G as well and do not contain $s \in A$, they have cardinality 3 or 4 each. If two of them would intersect, then the cardinalities of their complements would sum up to at most 5 by Lemma 3, and thus one of the complements would be a fragment of G of cardinality 1 or 2 which does not contain s, a contradiction. Hence $G - A$ has $2k$ *disjoint* fragments of cardinality 3 or 4 each which do not contain s, and, thus, each of them contains an end of G of cardinality 3 or 4. Hence we found $2k$ disjoint ends of cardinality 3 or 4 in G. By Claim 1, each of these must contain three vertices of $N(A)$, and, thus, $n = |N(A)| \geq 2k \cdot 3 = 6k$ – a contradiction. This proves Claim 2.

By Theorem 29, there exists a fragment A of cardinality 1 or 2 in G, and by Claim 2, $A = \{s\}$ is the *only* fragment of cardinality 1 or 2 in G.

Claim 3. Let B be an end of G of cardinality 3 or 4. Then $B \subseteq N(s)$.

Let us assume, to the contrary, that there exists a $y \in B - N(s)$.

By assumption and Claim 2, G does neither contain two vertices of degree n nor a fragment of cardinality 2. Hence, by Lemma 31, $G - B$ must be an $(n-|B|, k-1)$-graph.

Since $B \not\subseteq N(s)$, $A = \{s\}$ is not a fragment of $G-B$. Hence, by assumption and Claim 2, $G - B$ has no fragment of cardinality 1 or 2 at all. By Theorem 10, $G - B$ has $2k$ distinct ends, all of cardinality 3 or 4. By Lemma 17, they are all disjoint. By Lemma 1, they intersect $N(B)$ in at least three vertices each, implying that $n = |N(B)| \geq 6k$, a contradiction.

This proves Claim 3.

Finally, let us consider the $(n-1, k-1)$-graph $G-s$. By Theorem 10, $G-s$ has $2k$ distinct ends. Since they do not contain s, they must have cardinalities 3 or 4. As above, this leads to $2k$ *disjoint* ends of G of cardinality 3 or 4, and each of these intersects $N(s)$ in three vertices by Claim 3, implying that $n = |N(s)| \geq 6k$, again a contradiction. ∎

Another possibility to guarantee the existence of two disjoint fragments in an (n,k)-graph is to put restrictions not on its order but to strengthen the restriction on the ratio of n and k: Let G be an (n,k)-graph such that $n < \frac{24}{5}k - \frac{19}{5}$, i.e. $n - 1 < \frac{24}{5}(k-1)$. By Theorem 20, an atom A of G has cardinality 1 or 2, and, thus, $n - |A| \leq n - 1 < \frac{24}{5}(k-1)$. Again by Theorem 20, the $(n-|A|, k-1)$-graph $G - A$ has a fragment of cardinality 1 or 2, too (which is a fragment of G as well and disjoint from A). Therefore we proved that every (n,k)-graph with $n < \frac{24}{5}k - \frac{19}{5}$ has two disjoint fragments of cardinality 1 or 2. Taking this argument and Conjecture 13 into account, one could expect that the bound to n in that statement can be improved. Indeed, as the proof of the third theorem of this section will illustrate, the methods developed in [11] lead to an improvement of $\frac{9}{5}$ for the constant part of the bound $\frac{24}{5}k - \frac{19}{5}$.

Theorem 33. *Every (n,k)-graph with $n < \frac{24}{5}k - 2$ has two disjoint fragments of cardinality 1 or 2 each.*

Proof. Let G be an (n,k)-graph and $n < \frac{24}{5}k - 2$. If $k = 1$ then $n \leq 2$, and G has two disjoint fragments of cardinality 1 by Theorem 14. If $k = 2$ then $n \leq 7$, and G has two disjoint fragments of cardinality 1 or 2 each by Theorem 16 (or, alternatively, by Theorem 35 in the next section). Hence we may assume that $k \geq 3$ holds.

Let A be an atom of G. By Theorem 20, $|A| \in \{1, 2\}$. Let us assume, to the contrary, that G does not have two disjoint fragments of cardinality 1 or 2 each. By Lemma 30 there is a vertex $s \in A$ which is contained in every fragment of cardinality 1 or 2 in G. Since $k \geq 3$, $n-|A| \leq n-1 < 6\cdot(k-1)$; since $G-A$ is an $(n-|A|,k-1)$-graph, $G-A$ must have a fragment B of cardinality at most 3 by Theorem 21. By our assumption it follows that B is a proper end of G of cardinality 3. Then 3. of Lemma 31 does not hold obviously, and 2. does not hold either, for otherwise $N(B)$ contained two fragments of cardinality 1, contradicting our assumption. Hence $G-B$ is an $(n-3,k-1)$-graph. In particular, if F is a T-fragment of $G-B$ then F is a $T\cup B$-fragment of G.

Claim. Every fragment of cardinality 1 or 2 of G that intersects $N(B)$ must be contained in $N(B)$.

This is trivial for fragments of cardinality 1. Consider a fragment $\{x,y\}$ of G of cardinality 2 of G such that $x \in N(B)$. Note that x,y have at least $n-2$ common neighbors. If $y \in B$ then y would not be adjacent to any neighbor of x in $\overline{B}$, so y had degree n, contradicting the fact that B is an end. If $y \in \overline{B}$ then y would not be adjacent to any neighbor of x in B, so x had only one neighbor b in B; but then $B-\{b\}$ would be a fragment contained in the end B, which is again a contradiction. So $y \in N(B)$, which proves the claim.

Since $n-3 < \frac{24}{5}k-5 = \frac{24}{5}(k-1)-\frac{1}{5} < \frac{24}{5}(k-1)$, $G-B$ has a fragment of cardinality 1 or 2 by Theorem 20. This fragment must be contained in $N(B)$ by our claim, and it contains s. Therefore, by Lemma 30, *every* fragment of G of cardinality 1 or 2 intersects $N(B)$ and thus must be contained in $N(B)$ by our claim.

Hence $N(B)$ is an end cover of $G-B$ such that any end F of $G-B$ of cardinality at least 3 intersects it in at least 3 vertices (this is clear if $F \subseteq N(B)$, otherwise $F\cap\overline{B} \neq \emptyset$, and $|F\cap N(B)| \geq |B\cap N(F)| = 3$ follows from Lemma 1) and such that any fragment of $G-B$ of cardinality 1 or 2 intersects $N(B)$ in the vertex s. By Lemma 3, there is exactly one *end* A of $G-B$ of cardinality 1 or 2.

We construct a hypergraph H as follows: Take $N(B)-A$ plus three new vertices a,b,c as the vertex set of H, and let $\{a,b,c\}$ plus the intersections of $N(B)$ with each end of $G-B$ distinct from A form the edges of H. Since A does not intersect any other end of $G-B$ by the first part of Lemma 3, $|e| \geq 3$ follows for all $e \in E(H)$. By Lemma 17, H has a set of at least $\frac{|E(H)|}{2}$ independent edges. By Theorem 18, there exists a transversal of H of cardinality at most $\frac{5}{12}|V(H)| \leq \frac{5}{12}(n+2)$. Since a minimal transversal of H corresponds in a canonical way to an end cover of $G-B$ of the same cardinality (replace the unique vertex of the transversal which intersects the edge $\{a,b,c\}$ by s) and since, by Theorem 10, an end cover has cardinality at least $2\cdot(k-1)+2=2k$, $2k\leq\frac{5}{12}(n+2)$ follows, implying that $n\geq\frac{24}{5}k-2$, a contradiction. ∎

6. The small cases

As it is indicated in [10], we can not expect to obtain much structural information about a $(2,1)$-graph. On the other hand, it is possible to characterize the $(4,2)$-graphs, and they turn out to be "highly symmetric" [14] (cf. Theorem 25). As we know from Theorem 24, there is only one $(6,3)$- and only one $(8,4)$-graph, and both have nice symmetry properties. Hence one has the impression that the condition of being a $(2k,k)$-graph becomes more and more restrictive the larger k is.

Indeed, to handle the case $k\in\{5,6,7,10\}$ in the proof of Mader's conjecture, we need some further improvement to the number of disjoint small fragments in certain $(13,3)$-, $(9,2)$-, $(8,2)$-, and $(7,2)$-graphs. There are alternative proofs for the uniqueness of $(10,5)$-, $(12,6)$-, and $(14,7)$-graphs which do not rely on these particular results but on Theorem 26. However, the proof effort does not really decrease and the approach presented here fits better in the general setting of the next section than its alternatives.

The reader who is not so interested in these small cases is invited to skip the rather particular theorems of this section, and be ensured that our further arguments are *non-inductive*, i.e. the uniqueness for some $(2k,k)$-graph does not depend of the uniqueness for some $(2\ell,\ell)$-graph, $3\leq\ell<k$.

We start with a lemma on ends of cardinality 3 in an $(n,2)$-graph. Its proof can be slightly simplified by using the concept of generalized critical connectivity as introduced in [16]. However, I feel that a "direct" proof increases readability here.

Lemma 34. *Let C be an end of an $(n,2)$-graph G with $|C|=3$ such that $G-C$ is not an $(n-3,1)$-graph and such that $N(C)$ contains precisely two vertices $a\neq b$ of degree n in G. Then $G-C$ has connectivity $n-3$ (so every fragment of $G-C$ is a fragment of G, too) and one of the following is true.*

1. *There exists a fragment $D\neq\{a,b\}$ of $G-C$ such that $2\leq|D|\leq\frac{n-3}{2}$ or*
2. *There exists a fragment $D\subseteq N(a)\cap N(b)$ of $G-C$ such that $2\leq|D|\leq(|V(G)|-n)/2$. (In particular, $a,b\in N(D)$ holds.)*

Proof. Consider the set $X:=(V(G)-C)-(N(a)\cap N(b))$ and take $x\in X$. By Lemma 31, $C-N(a)$ and $C-N(b)$ are non-empty. Hence there exists a vertex $y\in C$ such that $\{x,y\}\not\subseteq N(a)$ and $\{x,y\}\not\subseteq N(b)$. Since G is an $(n,2)$-graph, $\{x,y\}\subseteq T$ for some smallest separating set T of G. If $F\cap C\neq\emptyset$ for some

T-fragment F then, by Lemma 2, $\overline{F}\cap\overline{C}=\emptyset$ and $2\geq|C\cap T|>|\overline{F}\cap N(C)|\geq 1$ – implying that $\overline{F}\cap C=\emptyset$ and so $\overline{F}\subseteq N(C)$ and $|\overline{F}|=1$. Since $T\neq N(a)$ and $T\neq N(b)$, $\overline{F},\{a\},\{b\}$ are disjoint fragments of cardinality 1 each, hence $N(C)$ contains three vertices of degree n, a contradiction. It follows that $F\cap C=\emptyset$ for every T-fragment F, implying that $\{x\}\cup C\subseteq T$.

Since $a,b\in X\neq\emptyset$ we proved that $G-C$ has connectivity $n-3$, and that every vertex in X is contained in some smallest separating set of $G-C$.

Among all fragments of G–C whose neighborhood intersects X we choose D of minimal cardinality. Clearly, $|D|\leq(|V(G)|-n)/2$. Since $\{a\},\{b\}$ are not fragments of $G-C$ we have $|D|\geq 2$ by the condition that there are no vertices of degree n in $N(C)-\{a,b\}$. If $|D|=2$ then $D\neq\{a,b\}$ (for if $D=\{a,b\}$ then a must be adjacent to every vertex in $N(D)-C$ since a is not adjacent to every vertex in C, and so must b; but then $N(D)$ can not contain a vertex of X, a contradiction to the choice of D).

If $X\cap D=\emptyset$ then $D\subseteq N(a)\cap N(b)$, and 2. holds.

Otherwise, there exists a vertex $x\in X\cap D$. There exists a smallest separating set T of $G-C$ containing x. We show that $|D|\leq\frac{n-3}{2}$ to accomplish the proof. (This would follow from Theorem 1 in [16], applied to $G-C$ with $\mathcal{S}:=\{\{x\}:x\in X\}$.)

Let's assume for a while that there exists a T-fragment F intersecting D. By Lemma 1, $|F\cap N(D)|>|\overline{D}\cap T|$ and $|D\cap T|>|\overline{F}\cap N(D)|$, for otherwise $F\cap D$ would be a fragment such that $x\in N(F\cap D)=(F\cap N(D))\cup(T\cap N(D))\cup(T\cap D)$, contradicting the minimality of $|D|$. By Lemma 1, $\overline{F}\cap\overline{D}=\emptyset$.

If $\overline{F}\cap D\neq\emptyset$ then, analogously, $F\cap\overline{D}=\emptyset$. By Lemma 1, $|D|>|D\cap T|>|\overline{F}\cap N(D)|\geq|\overline{D}\cap T|=|\overline{D}|$. Since $\overline{D}$ is also a fragment of $G-C$ whose neighborhood intersects X, the minimality of $|D|$ is again violated. Hence $\overline{F}\subseteq N(D)$, and, thus, $|\overline{F}|=|\overline{F}\cap N(D)|<|D\cap T|<|D|$, which is absurd, too.

Hence $D\subseteq T$. If $F\subseteq N(D)$ for some T-fragment F then $|F\cap N(D)|=|F|\geq|D|$ by choice of D, otherwise $F\cap\overline{D}\neq\emptyset$, and $|F\cap N(D)|\geq|D\cap T|=|D|$ follows from Lemma 1. In either case, $|F\cap N(D)|\geq|D|$, and, by symmetry, $|\overline{F}\cap N(D)|\geq|D|$, implying that $|D|\leq(|N(D)-T|)/2\leq(n-3)/2$. ∎

Now we are prepared to refine the theorems of the preceding section for certain small values of n and k.

Theorem 35. *Every $(n,2)$-graph with $n\leq 9$ has two disjoint fragments of cardinality 1 or 2 each.*

Proof. Suppose, to the contrary, that there exists an $(n,2)$-graph G with $n\leq 9$ without two disjoint fragments of cardinality 1 or 2 each.

Claim 1. *G has no ends of cardinality 3, and if G has an end of cardinality 4 then G has a fragment of cardinality 2, too.*

For suppose that B is an end of G of cardinality 3 or 4. If $G-B$ is an $(n-|B|,1)$-graph then $G-B$ has four disjoint fragments. By assumption, at most one of these is a fragment of cardinality 1 or 2, and, therefore, at least 3 of them have cardinality at least 3 and, therefore, have to contain at least three vertices of $N(B)$ by Lemma 1. It follows $n=|N(B)|\geq 1+3\cdot 3=10$, which is absurd. Hence $G-B$ is not an $(n-|B|,1)$-graph. Since G does not contain two vertices of degree n by assumption, it follows that 3. of Lemma 31 holds. This proves Claim 1.

Claim 2. *G has no fragment of cardinality 2.*

For suppose that G had a fragment A of cardinality 2. By assumption, A does not consist of two vertices of degree n in G, and hence $G-A$ is an $(n-2,1)$-graph by 1. of Lemma 6. By Theorem 14, $G-A$ has a fragment F of cardinality at most 3. F contains an end B of G, and by Claim 1, $|B|\leq 2$. Hence A,B are disjoint fragments of cardinality 1 or 2 each, contradicting our assumption. This proves Claim 2.

By Claim 1 and Claim 2 we know that G has no ends of cardinality 3 or 4 (or 2). Take $x\in V(G)$. Since $G-x$ is an $(n-1,1)$-graph, it must have two disjoint ends of cardinality at most $\frac{n-1}{2}\leq\frac{8}{2}=4$ each. Thus, G has two disjoint fragments of cardinality at most 4 and, therefore, two disjoint ends of cardinality 1 or 2 each – a contradiction. ∎

Theorem 36. *Every $(n,3)$-graph with $n\leq 13$ has two disjoint fragments of cardinality 1 or 2 each.*

Proof. Suppose, to the contrary, that there exists an $(n,3)$-graph G with $n\leq 13$ without two disjoint fragments of cardinality 1 or 2 each.

Claim 1. *G has no fragment of cardinality 2.*

If C is a fragment of cardinality 2 then, by assumption, it is not the disjoint union of two fragments of cardinality 1. Hence $G-C$ is an $(n-2,2)$-graph by 1. of Lemma 6. Take an atom B of $G-C$. By assumption, $|B|\geq 3$, and $(G-C)-B$ is an $(n-2-|B|,1)$-graph whose fragments intersect $N(B)$ in at least 3 vertices by Lemma 1. Since $(G-C)-B$ has four disjoint fragments by Theorem 8, $n-2=|N(B)-C|\geq 3\cdot 4$ follows, violating $n\leq 13$. This proves Claim 1.

Claim 2. *G has no end of cardinality 3 or 4.*

Let C be an end of G with $|C|\in\{3,4\}$. By Claim 1, the assumption, and Lemma 31, $G-C$ is an $(n-|C|,2)$-graph and must have six disjoint fragments by Theorem 9. At most one of them has cardinality 1, and therefore at least five of them intersect $N(C)$ in at least 3 vertices by Lemma 1. It follows that $13\geq n=|N(C)|\geq 1+5\cdot 3\geq 16$, a contradiction. This proves Claim 2.

By Theorem 20 and Claim 1, G contains a vertex a of degree n. Let B be an atom of the $(n-1,2)$-graph $G-a$, and let C be an atom of the $(n-1-|B|,1)$-graph $(G-a)-B$. By Claim 1, the assumption, and Claim 2, $|B|\geq 5$ and $|C|\geq 5$ follow. On the other hand we know from Theorem 14 that $|C|\leq \frac{n-1-|B|}{2}\leq \frac{7}{2}$, a contradiction. ∎

(Parts of the proof of Theorem 36 mimic the method of *atomic sequences*, as described in [4].) We can improve the estimates on the number of disjoint fragments of cardinality 1 or 2 in Theorem 36 if we add a further restriction on the order of G:

Theorem 37. *Every $(n,3)$-graph of order at most $n+7$ and with $n\leq 13$ has three disjoint fragments of cardinality* 1 *or* 2 *each.*

Proof. Let's assume, to the contrary, that G is an $(n,3)$-graph of order at most $n+7$ and with $n\leq 13$ which does *not* have three disjoint fragments of cardinality at most 1 or 2 each.

Claim 1. If A is a fragment of cardinality 2 then A consists of two adjacent vertices of degree 2.

For otherwise, $G-A$ is an $(n-2,2)$-graph of order at most $n-2+7$ by 1. of Lemma 6. By Theorem 32, $G-A$ has two disjoint fragments of cardinality 1 or 2 each, which form together with A a system of three disjoint fragments of cardinality 1 or 2 each – a contradiction. This proves Claim 1.

By Theorem 10 and Lemma 17 (or by a simple induction argument), G has four disjoint proper ends. Since proper ends of G have cardinality 1, 2, or 3, it follows by assumption that G has an end C of cardinality 3. If $G-C$ is an $(n-3,2)$-graph then it would have six disjoint fragments by Theorem 9; since $|N(C)|=n\leq 13$, at least three of these fragments intersect $N(C)$ in less than three vertices; from Lemma 1 it follows that these three fragments are contained in $N(C)$ and are fragments of cardinality at most 2 – a contradiction.

Hence we may assume that $G-C$ is not an $(n-3,2)$-graph. By Lemma 31 we know that $N(C)$ contains two vertices $a\neq b$ of degree n which are adjacent to precisely two vertices in C each. We claim that $G-(\{a\}\cup C)$ is an $(n-4,1)$-graph. To see this, take $x\in V(G)-(\{a\}\cup C)$; we have to find a smallest separating set $T\supseteq \{x,a\}\cup C$ of G. Since G is an $(n,3)$-graph, there exists a smallest separating set T containing x,a and the vertex $c\in C-N(b)$. To show that, indeed, $C\subseteq T$, let's suppose, to the contrary, that $F\cap C\neq\emptyset$ for some T-fragment F. By Lemma 2, $2\geq |C\cap T|>|\overline{F}\cap N(C)|\geq 1$, and, thus, $\overline{F}\subseteq N(C)$ and $|\overline{F}|=1$. Since $\overline{F}\notin \{\{a\},\{b\}\}$, G contains three vertices of degree n, a contradiction.

By Theorem 8, the $(n–4,1)$-graph $G–(\{a\}∪C)$ has four disjoint fragments F_1, F_2, F_3, F_4. By Claim 1, by our assumption, and by the fact that $\{b\}$ is not a fragment of $G-C$ and therefore of $G-(\{a\}\cup C)$ neither, none of the fragments of $G-(\{a\}\cup C)$ has cardinality 1 or 2. It follows that $|F_i \cap (N(C)-\{a\})| \geq 3$ for all $i \in \{1,2,3,4\}$ and, since $|N(C)-\{a\}| = n-1 \leq 12$, $|F_i \cap (N(C)-\{a\})| = 3$. Hence $b \in F_i$ for some $i \in \{1,2,3,4\}$, and $\overline{F_i}$ has cardinality 3 or 4. By Claim 1 and since $a \in N(F_i)$ and $b \in F_i$, $\overline{F_i}$ contains an end B of G of cardinality 3 or 4. By Claim 1 and since $b \in \overline{B}$, $N(B)$ does neither contain two vertices of degree n nor a fragment of cardinality 2. Hence, by KuTheoremLemma18, $G-B$ is an $(n-|B|,2)$-graph which must have six disjoint fragments by Theorem 9. Since $|N(B)| = n \leq 13$, at least three of these intersect $N(B)$ in less than 3 vertices, implying as in the proof of Claim 1 that they form a system of disjoint fragments contained in $N(B)$ – a contradiction. ∎

Theorem 38. *Every $(n,2)$-graph with $n \leq 7$ has three disjoint fragments of cardinality 1 or 2 each.*

Proof. Suppose, to the contrary, that there exists an $(n,2)$-graph G with $n \leq 7$ without three disjoint fragments of cardinality 1 or 2 each.

Claim 1. If A is a fragment of cardinality 2 then it consists of two adjacent vertices of degree n each.

For otherwise, $G-A$ is an $(n-2,1)$-graph by 1. of Lemma 6 which has, by Theorem 14, two disjoint fragments of cardinality at most $\frac{n-2}{2} \leq \frac{5}{2} < 3$. Together with A, they form a system of three disjoint fragments of cardinality 1 or 2 each, violating our assumption. This proves Claim 1.

Claim 2. G has no end of cardinality 3.

For suppose that C is an end of cardinality 3. If $G–C$ is an $(n–3,1)$-graph then it has four disjoint fragments by Theorem 8. Since $|N(C)| = n \leq 7$, at least three of them intersect $N(C)$ in less than three vertices. By Lemma 1, they must be contained in $N(C)$ and therefore be fragments of cardinality 1 or 2 each, violating our assumption. Hence $G-C$ is not an $(n-3,1)$-graph, and, therefore, 2. of Lemma 31 holds. It follows that $N(C)$ contains vertices $a \neq b$ of degree n in G, and there are no further vertices of degree n in $N(C)$ by assumption. Consequently, there exists a fragment D of $G–B$ as in 1. or 2. of Lemma 34. If 1. applies then $|D| \leq 2$ and $D \neq \{a,b\}$ follow, implying that there must be a vertex of degree n in $D-\{a,b\}$ by Claim 1, a contradiction. If 2. applies then $|D| \geq 3$ by Claim 1. By Lemma 1, $|N(C) \cap D| \geq 3$, and, since $a, b \in N(C)$, $|\overline{D} \cap N(C)| \leq 2$, implying that $\overline{D} \subseteq N(C)$; hence $\overline{D}$, $\{a\}$, $\{b\}$ are disjoint fragments of cardinality 1 or 2 each, a contradiction. This proves Claim 2.

By Theorem 20 (or Theorem 35) and Claim 1, G contains a vertex x of degree n. By Theorem 14, the $(n-1,1)$-graph $G-x$ has two disjoint fragments of cardinality at most $\frac{n-1}{2} \leq 3$, and these contain two disjoint ends of G disjoint from $\{x\}$. By Claim 2, these ends have cardinality 1 or 2 each, and they form, together with $\{x\}$, a system of three disjoint fragments of cardinality 1 or 2 each – a contradiction. ∎

Theorem 39. *Every* $(n,2)$*-graph of order at most* $n+7$ *and with* $n \leq 8$ *has three disjoint fragments of cardinality* 1 *or* 2 *each.*

Proof. Suppose, to the contrary, that there exists an $(n,2)$-graph G with $n \leq 8$ and $|V(G)| \leq n+7$ without three disjoint fragments of cardinality 1 or 2 each.

Claim 1. If A is a fragment of cardinality 2 then it consists of two adjacent vertices of degree n each.

For if A is a fragment of cardinality 2 which does not consist of two adjacent vertices of degree n each then $G-A$ is an $(n-2,1)$-graph by 1. of Lemma 6. It has four disjoint fragments F_1, F_2, F_3, F_4 by Theorem 8.

Case 1. $|F_i \cap N(B)| \geq 2$ for all $i \in \{1,2,3,4\}$. Since $|N(B)| \leq 8$, $|F_i \cap N(B)| = 2$ holds for all $i \in \{1,2,3,4\}$. By assumption, $F_i \cap \overline{A} = \emptyset$ for at most one $i \in \{1,2,3,4\}$. Since $|\overline{A}| \leq 5$, there are $i \neq j$ in $\{1,2,3,4\}$ such that $1 \leq |F_i \cap \overline{A}| \leq 2$ and $1 \leq |F_j \cap \overline{A}| \leq 2$. By Lemma 1, $F_i \cap \overline{A}$ and $F_j \cap \overline{A}$ are fragments of G, and thus they form together with A a system of three disjoint fragments of cardinality 1 or 2 each.

Case 2. $|F_i \cap N(B)| = 1$ for some $i \in \{1,2,3,4\}$. By Lemma 1, F_i is a fragment of G of cardinality 1, and, by assumption, $|F_j \cap N(B)| \geq 2$ for all $j \in \{1,2,3,4\} - \{i\}$. Since $|N(B)| \leq 8$, there are $j \neq k$ in $\{1,2,3,4\} - \{i\}$ such that $|F_j \cap N(B)| = |F_k \cap N(B)| = 2$ holds. By assumption, $F_j \cap \overline{A}$ and $F_k \cap \overline{A}$ are not empty, and, by Lemma 1, they are both fragments. Since $|\overline{A}| \leq 5$, one of $F_j \cap \overline{A}$, $F_k \cap \overline{A}$ is a fragment of cardinality at most 2 and forms together with A and F_i a system of three disjoint fragments of cardinality 1 or 2 each.

Hence in either case, we found a contradiction. This proves Claim 1.

Claim 2. If B is an end of G of cardinality 3 then its neighborhood contains two vertices of degree n. If $G-B$ is an $(n-3,1)$-graph then they are both adjacent to all vertices of B, and otherwise they are both adjacent to precisely two vertices of B.

Suppose that $G–B$ is an $(n–3,1)$-graph. By Theorem 14, $G–B$ must have two fragments A, C of cardinality at most $\frac{n-3}{2} \leq \frac{5}{2} < 3$. If A has cardinality 2 then it is the union of two fragments of G of cardinality 1 by Claim 1 which form, together with C, a system of three disjoint fragments of cardinality 1 or 2 each, a contradiction. Hence we may assume $|A|=1$ and, by symmetry, $|C|=1$. Hence each of A, C consists of a vertex of degree n in G adjacent to all vertices of B.

Suppose now that $G-B$ is *not* an $(n-3,1)$-graph. Then 2. of Lemma 31 holds: there are two vertices of degree n in $N(B)$ which are adjacent to precisely two vertices of B each.

In either case, Claim 2 is proved.

Claim 3. If C is an end of cardinality 3 then its neighborhood contains two vertices of degree n which are both adjacent to all vertices of B.

Otherwise, $G–C$ is not an $(n–3,1)$-graph and $N(C)$ contains two vertices $a\neq b$ of degree n such that $C\nsubseteq N(a)$ and $C\nsubseteq N(b)$ by Claim 2.

By assumption, $a\neq b$ are the *only* vertices of degree n in G. Hence there exists a fragment D as in 1. or in 2. of Lemma 34. If $|D|=2$ then $D=\{a,b\}$ by Claim 1 and the assumption, and so neither 1. nor 2. of Lemma 34 can hold. Hence $|D|\geq 3$. Since $n\leq 8$, 2. of Lemma 34 holds, and since $|V(G)|\leq n+7$, $|D|=3$ and $D\in N(a)\cap N(b)$ follow. By Lemma 1, $D\subseteq N(C)$. Since $a,b\in N(C)\cap N(D)$, it follows $|\overline{C}\cap N(D)|\leq 3$ and $|\overline{D}\cap N(C)|\leq 3$.

If $\overline{C}\cap\overline{D}\neq\emptyset$ then $3\geq|\overline{C}\cap N(D)|\geq|D\cap N(C)|=3$ by Lemma 1, implying that $\overline{C}\cap\overline{D}$ is a fragment distinct from $\{a\},\{b\}$ of cardinality at most $|\overline{C}|-|\overline{C}\cap N(D)|\leq 4-3=1$, violating our assumption.

Hence $\overline{C}\cap\overline{D}=\emptyset$, and, thus, $\overline{C}$ and $\overline{D}$ are fragments of cardinality 3, too. Since $a,b\in N(C)\cap N(D)$ it follows by Claim 1 and the assumption that $C,\overline{C},D,\overline{D}$ are disjoint *ends* of G of cardinality 3. By Claim 2, a has at least two neighbors in each of $C,\overline{C},\overline{D}$. Since a is adjacent to every vertex in D, it must have at least 9 neighbors, violating the fact that a is a vertex of degree $n\leq 8$.

This proves Claim 3.

Claim 4. G has an end of cardinality 3.

By Theorem 20, G has a fragment of cardinality 1 or 2. By Claim 1, G must contain a vertex a of degree n. The $(n-1,1)$-graph $G-a$ has two disjoint fragments of cardinality at most $\frac{n-1}{2}=\frac{7}{2}<4$. Not both of them contain a vertex of degree n by assumption, and none of them can contain a fragment of cardinality 2 by assumption and Claim 1. Thus, one of them must be an end of G of cardinality 3, which proves Claim 4.

To finish the proof, we consider an end C of cardinality 3, which exists by Claim 4. We would like to apply Lemma 31. By Claim 2, 2. of Lemma 31 does not hold, and, clearly, 3. of Lemma 31 does not hold. Hence 1. of Lemma 31 holds, so $G-C$ is an $(n-3,1)$-graph. By Theorem 8, $G-C$ has four disjoint fragments F_1,F_2,F_3,F_4. Two of them must intersect $N(C)$ in less than three vertices. Without loss of generality, let these be F_1 and F_2. By Claim 1 and Lemma 1, $|F_1|=|F_2|=1$, and by assumption and Lemma 1, $|F_3\cap N(C)|=|F_4\cap N(C)|=3$ follows.

If $F_3\cap\overline{C}\neq\emptyset$ then we obtain by Lemma 1 that $F_3\cap\overline{C}$ is a fragment, since $|F_3\cap N(C)|=|C\cap N(F_3)|=3$. By assumption, $F_3\cap\overline{C}$ has cardinality at least 3 and, since $4\geq|\overline{C}|\geq|\overline{C}\cap F_3|+|\overline{C}\cap N(F_3)|\geq|\overline{C}\cap F_3|+|\overline{F_3}\cap N(C)|\geq 3+1$, it must have cardinality exactly 3.

If $F_4\cap\overline{C}\neq\emptyset$ then, by symmetry, $F_4\cap\overline{C}$ is a fragment of cardinality 3. Consequently, by Claim 1, we find always two disjoint ends of cardinality 3 among $F_3,F_4,F_3\cap\overline{C},F_4\cap\overline{C}$. Taking C into account, there exist three disjoint ends of cardinality 3, which is absurd, since, by Claim 3, the vertex a in F_1 must be adjacent to every vertex in each of these fragments, but has a degree $n\leq 8$. ∎

Theorem 39 is best possible in the sense that an $(8,2)$-graph on at most $8+7$ vertices has not *more* than three disjoint fragments of cardinality 1 or 2 in general (as it is indicated by the $(8,2)$-graph of order $8+4$ obtained from $K_8-(1-factor)$ by replacing a triangle Δ by a K_9 and all edges between the new K_9 and the neighbors of Δ, cf. the graph $G(2,1,3)$ in Section 4 of [11]).

This improves if we sharpen the connectivity and the order once more.

Theorem 40. *Every $(n,2)$-graph of order at most $n+6$ and with $n\leq 7$ has four disjoint fragments of cardinality 1 or 2 each.*

Proof. Suppose, to the contrary, that there exists an $(n,2)$-graph G with $n\leq 7$ and $|V(G)|\leq n+6$ which has no system of four disjoint fragments of cardinality 1 or 2 each.

Claim 1. If A is a fragment of cardinality 2 then A consists of two adjacent vertices of degree n.

For otherwise, $G-A$ is an $(n-2,1)$-graph by 1. of Lemma 6. It has four disjoint fragments F_1,F_2,F_3,F_4. Since $|N(A)|=n\leq 7$, we have $|N(A)\cap F_i|=1$ and thus $|F_i|=1$ for some $i\in\{1,2,3,4\}$. Without loss of generality, $i=1$, so $|F_1|=1$.

If $|F_i\cap N(A)|\geq 2$ for all $i\in\{2,3,4\}$ then $|F_i\cap N(A)|=2$ for all $i\in\{2,3,4\}$. Since $|\overline{A}|\leq 4$, there are $i\neq j$ in $\{2,3,4\}$ such that $|F_i\cap\overline{A}|\leq 2$ and $|F_j\cap\overline{A}|\leq 2$. If $F_i\cap\overline{A}$ is empty, then F_i is a fragment of cardinality 2, otherwise $F_i\cap\overline{A}$ is a fragment of cardinality 1 or 2 by Lemma 1. Hence, in either case, there are two disjoint fragments among $F_i,F_j,F_i\cap\overline{A},F_j\cap\overline{A}$ which form, together with A and F_1, a system of four disjoint fragments of G of cardinality 1 or 2 each, a contradiction.

Hence $|F_i|=1$ for some $i\in\{2,3,4\}$. Without loss of generality, $i=2$, so $|F_2|=1$. We may assume that $|F_3\cap N(B)|\geq 2$ and $|F_4\cap N(B)|\geq 2$, for otherwise A, F_1, F_2, F_3 or A, F_1, F_2, F_4 would be a system of four disjoint fragments of cardinality 1 or 2 each by Lemma 1. If we had $|F_3 \cap N(B)| = |F_4 \cap N(B)| = 2$ then either one of F_3, F_4 or one of $F_3 \cap \overline{A}$, $F_4 \cap \overline{A}$ would be a fragment of cardinality 1 or 2, which then forms together with A, F_1, F_2 a system of three disjoint fragments of cardinality 1 or 2 each, a contradiction.

Hence we may assume, without loss of generality, that $|F_3 \cap N(B)| = 2$ and $|F_4 \cap N(B)| = 3$. If $|F_3 \cap \overline{A}| \leq 2$ then one of $F_3, F_3 \cap \overline{A}$ would be a fragment of cardinality 1 or 2, which is again absurd. Hence $|F_3 \cap \overline{A}| \geq 3$, and since $|\overline{A}| \leq 4$ and $N(F_3) \cap \overline{A} \neq \emptyset$ it follows $|F_3 \cap \overline{A}| = 3$, implying that F_3 is the complement of some fragment of cardinality 1 of $G - A$, which is, without loss of generality, F_1. If the fragment $F_3 \cap \overline{A}$ is not an end of G then it contains a fragment of cardinality 1 or 2, which is impossible. Hence $B := F_3 \cap \overline{A}$ is an end of cardinality 3. If $G - B$ is an $(n-3,1)$-graph then by Theorem 14 there must be two disjoint fragments of cardinality 1 or 2 each, being contained in $N(B)$ by Lemma 1, which form together with F_1 and A a system of four disjoint fragments of G of cardinality 1 or 2 each, a contradiction. If $G - B$ is not an $(n-3,1)$-graph then it follows from 2. of Lemma 31 that there are two vertices of degree n in $N(B)$, which is absurd as well.

This proves Claim 1.

Claim 2. If $|V(G)| = n+6$ then there exists no end of cardinality 4.

For suppose, to the contrary, that there exists an end B of cardinality four with $|\overline{B}| = 2$. If $G - B$ is an $(n-4,1)$-graph then it has two disjoint fragments of cardinality at most $\frac{n-4}{2} \leq \frac{3}{2}$, by Theorem 14 and hence there are two vertices of degree n in $N(B)$. Otherwise, 2. or 3. of Lemma 31 hold. If 2. holds then there are two vertices of degree n in $N(B)$, and if 3. holds then there is a fragment F of cardinality 2 contained in $N(B)$; by Claim 1, F consists of two vertices of degree n. Thus, in either case, $N(B)$ contains two vertices of degree n. Since $|\overline{B}| = 2$, $\overline{B}$ consists of two vertices of degree n by Claim 1, and thus there are four disjoint fragments of cardinality 1 in G, which is absurd. This proves Claim 2.

Claim 3. If C is an end of cardinality 3 then it is adjacent to three vertices of degree n. In particular it follows by Claim 1 and the assumption that $\overline{C}$ is an end of cardinality 3.

If $G - C$ is an $(n-3,1)$-graph then it has four disjoint fragments by Theorem 8. Since $|N(C)| \leq 7$, at most one of them intersects $N(C)$ in more than two vertices. Hence the other three are fragments of cardinality 1 or 2, and the assertion follows from Claim 1.

If $G - C$ is not an $(n-3,1)$-graph then there are two vertices $a \neq b$ of degree n in the neighborhood of G by Lemma 31. Let us assume that there is no further one. Then there exists a fragment D as in 1. or 2. of Lemma 34.

By assumption and Claim 1 we obtain $|D|\geq 3$, implying that 2. of Lemma 34 holds and thus $|D|=|\overline{D}|=3$. It follows $7\geq|N(C)|=|N(C)\cap D|+|N(C)\cap\overline{D}|$ $+\ |N(C)\cap N(D)|\geq 3+3+|\{a,b\}|\geq 8$ by Lemma 1, a contradiction.

This proves Claim 3.

Claim 4. There exists no end of cardinality 3.

For suppose that C is such an end. By Claim 3, $N(C)$ contains three vertices a,b,c of degree n and $\overline{C}$ must be an end of cardinality 3 as well. How does an end $D\notin\mathcal{F}:=\{\{a\},\{b\},\{c\},C,\overline{C}\}$ of G look like? By assumption, $|D|\geq 3$, and if $|D|=3$ then D is disjoint from any fragment of $\mathcal{F}$ by Lemma 17, and so is the fragment $\overline{D}$, which is an end of cardinality 3 by Claim 3. But then $|\bigcup\mathcal{F}\cup D\cup\overline{D}|=15>n+6\geq|V(G)|$, which is absurd. Hence $|D|\geq 4$. Since $|V(G)|=n+|C|+|\overline{C}|=n+6$ we know from Claim 3 that $|D|\neq 4$ holds, hence $|D|=5$.

It follows that every end not contained in $\mathcal{F}$ must have cardinality 5 and thus must be one of $\overline{\{a\}}$, $\overline{\{b\}}$, $\overline{\{c\}}$.

Since we can dominate the five ends in $\mathcal{F}$ by less than 6 vertices but every end cover of G has at least 6 vertices by Theorem 10, we may assume that, without loss of generality, $\overline{\{a\}}$ is an end. Then $b,c\in N(a)$ and either $C\not\subseteq N(a)$ or $\overline{C}\not\subseteq N(a)$. So we can dominate the six ends in $\mathcal{F}\cup\{\overline{\{a\}}\}$ by 5 vertices as well, implying that, without loss of generality, $\overline{\{b\}}$ is an end, too. This implies that a,b,c form a triangle. Since $a,b,c\in N(C)$ and $C,\overline{C}$ are ends, a,b,c each have at least two neighbors in each of $C,\overline{C}$. So b,c have at most one neighbor in $N(C)-\{a,b,c\}$ each, implying that at least one of the three or four vertices in $\overline{\{a\}}\cap N(C)$ is not adjacent to any of b,c (and a).

Take any $x\in V(G)-(N(a)\cup N(b)\cup N(c))\neq\emptyset$. By assumption, $G-x$ has no fragments of cardinality 1, and no fragments of cardinality 2 by Claim 1 and assumption. Hence $G-x$ can be partitioned into four disjoint fragments $D,F,\overline{D},\overline{F}$ of G of cardinality 3 by Theorem 8. If one of these is not an end of G then, by Claim 1, it must contain at least one vertex of degree n. Hence at least one of these must be an end of G of cardinality 3, and, as C and $\overline{C}$ are the unique ends of cardinality 3 in G, $F=C$ (and $\overline{F}=\overline{C}$) without loss of generality. Since D and $\overline{D}$ are not ends they both must contain vertices of degree n – but these are mutually adjacent, a contradiction.

This proves Claim 4.

By Claim 1 and Claim 4, every end of G has either cardinality 1, 4, or 5. If there is an end of cardinality 5 then, by Claim 2, there is no end of cardinality 4, and hence, in either case, every end of G has cardinality 1 or is the complement of an end of cardinality 1. Since G has six distinct fragments we find, by assumption, that there are precisely three ends F_1,F_2,F_3 of cardinality 1 in G, their complements are ends of cardinality $|V(G)|-n-1\in$

$\{4,5\}$, and beside these six ends there are no further ends. In both of the cases $|V(G)|-n=1+4$ or $|V(G)|-n=1+5$, two of the three non-trivial ends of G must intersect.

Thus we can dominate the non-trivial ends of G with two vertices and the others with 3 vertices, implying that G has an end cover of cardinality 5 – which contradicts Theorem 10. ∎

7. Covering all small fragments

The results of the two preceding sections are of the form "Every (n,k)-graph with n 'being small enough', compared with k, has a certain number of disjoint fragments of cardinality 1 or 2 each". The central theorem of this paper will enable us to use these results to estimate the number of disjoint small fragments in an $(n+1,k+1)$-, $(n+2,k+2)$-, ..., and, finally, in an $(2\ell-1,\ell-1)$-graph of a certain order.

For a graph G, let

$$f(G) := \min\{|S| : S \subseteq V(G) \text{ intersects every fragment of } G \text{ of cardinality 1 or 2}\}$$

be the smallest number of vertices by which the fragments of cardinality 1 or 2 can be covered. The following Theorem states that $f(G)$ is equal to the largest size of a system of disjoint fragments of cardinality 1 or 2 in G, and gives a lower bound to $f(G)$ for an $(n,1)$-graph G in terms of $f(G-x)$ for each $x \in V(G)$.

Theorem 41. *A graph G has a system of $f(G)$ disjoint fragments of cardinality 1 or 2 each but no larger such system, and if G is an $(n,1)$-graph then*

$$f(G) \geq \frac{|V(G)|}{n} \cdot \min\{f(G-x) : x \in V(G)\}.$$

Proof. Let G be a graph of connectivity n. Let $S \subseteq V(G)$ intersect every fragment of G of cardinality 1 or 2, and suppose that $|S| = f(G)$. For every $s \in S$, let us define $C(s) := \{s\} \cup \{t \in V(G) - S : \{s,t\} \text{ is a fragment of } G\}$.

Claim 1. $C(s) \cap C(t) = \emptyset$ for $s \neq t$ in S.

If two distinct fragments of cardinality two have a non-empty intersection then this intersection itself must be a fragment of cardinality 1 by Lemma 4 and thus must be contained in S. Thus, $C(s) \cap C(t) - S = \emptyset$ for $s \neq t$ in S, which proves Claim 1.

Claim 2. $|N(C(s))| = n$ for all $s \in S$.

Case 1. $|C(s)|=1$. If s had degree exceeding n then every fragment of cardinality 1 or 2 containing s would have cardinality 2 and would be contained in S as $C(s)-S=\emptyset$. Consequently, $S-\{s\}$ intersects every fragment of cardinality 1 or 2, which contradicts the minimality of S. Hence s has degree n, so $|N(C(s))|=|N(\{s\})|=n$, which settles Case 1.

Case 2. $|C(s)|=2$. Then $C(s)$ is a fragment by definition.

Case 3. $|C(s)|\geq 3$. Then $\{s\}$ must be a fragment by Lemma 4, so s has degree n. Every $t\in C(s)-\{s\}\subseteq V(G)-S$ has degree $n+1$, since $\{s,t\}$ is a fragment but $\{t\}$ is not (as $t\notin S$). If $x\in V(G)$ is adjacent to s then x must be adjacent to every $t\in C(s)-\{x\}$, since for every $t\in C(s)-\{s\}$, $\{s,t\}$ is a fragment such that every neighbor of s in $N(\{s,t\})$ is a neighbor of t as well. (In particular, $C(s)$ induces a complete subgraph of G.) If $x\neq s$ is adjacent to two distinct t,t' in $C(s)-\{s\}$ then $x\in N(\{s,t\})\cap N(\{s,t'\})$, implying that x is adjacent to s and thus to every vertex in $C(s)$. Hence $x\in C(s)\cup N(C(s))$ is either adjacent to every vertex in $C(s)-\{x\}$, or $x\in N(C(s))$ and x has a unique neighbor $\phi(x)$ in $C(s)$, which is distinct from s.

If we have $\phi(x)=\phi(y)$ then $x=y$, for otherwise $\{s,\phi(x)\}$ would be a fragment such that s was non-adjacent to two vertices in $N(\{s,\phi(x)\})$ (namely to x,y), which is impossible. Vice versa, for $t\neq t'$ in $C(s)-\{s\}$, $t\in N(\{s,t'\})$ must have a neighbor y in $\overline{\{s,t'\}}\cap N(\{s,t\})$; y is non-adjacent to s and thus $\phi(y)=t$ and $y\in N(C(s))$.

Hence for every $t\in C(s)-\{s\}$ there exists a unique $y=\phi^{-1}(t)\in N(C(s))$ such that $\phi(y)=t$. Since any vertex in $N(C(s))$ which has not a unique neighbor in $C(s)$ must be adjacent to s it follows that

$$\begin{aligned}|N(C(s))| &= |N(s)\cap N(C(s))| + |\phi^{-1}(C(s)-\{s\})|\\ &= |N(s)\cap N(C(s))| + |C(s)-\{s\}|\\ &= |N(s)\cap N(C(s))| + |N(s)\cap C(s)|\\ &= |N(s)| = n.\end{aligned}$$

This proves Claim 2.

If $|C(s)|\geq 2$ then $C(s)$ contains a fragment of cardinality 2 by definition, and if $|C(s)|=1$ then it is a fragment of cardinality 1 by Claim 2. Hence we may choose for each $s\in S$ a fragment of cardinality 1 or 2 contained in $C(s)$, and these fragments are disjoint by Claim 1. Therefore, there exists a system of $f(G)=|S|$ many disjoint fragments of cardinality 1 or 2 each, and clearly there is no larger such system. This proves the first statement of the Theorem.

Let G be an $(n,1)$-graph now, and let $m:=\min\{f(G-x):x\in V(G)\}$. We say that $x\in V(G)$ and $A\subseteq V(G)$ are *adjacent* if $x\in N(A)$.

Claim 3. Every vertex in G is adjacent to at least m of the $C(s)$ $(s \in S)$.

If $x \in C(s) - \{s\}$ then every neighbor of s in G distinct from x is a neighbor of x in G as well. Hence, if s is adjacent to at least m of the $C(s)$ then so is x by Claim 1. Therefore it suffices to prove that every $x \in V(G) - (\bigcup_{s \in S} C(s) - S)$ is adjacent to at least m of the $C(s)$.

Since G is an $(n,1)$-graph, every fragment of $G - x$ of cardinality 1 or 2 is a fragment of G of cardinality 1 or 2, too, and in case $x \in S$, these fragments are disjoint from $C(x)$. Let $T := \{s \in S : x \in N(C(s))\}$. Since every fragment of cardinality 1 or 2 of $G - x$ contains a vertex in T, $|T| \geq f(G - x) \geq m$ follows. This proves Claim 3.

By Claim 2, for every $s \in S$, $C(s)$ is adjacent to precisely n vertices of G. On the other hand, by Claim 3, every vertex in $V(G)$ is adjacent to at least m of the $C(s)$ $(s \in S)$. It follows that $|S| \cdot n \geq |V(G)| \cdot m$. ∎

8. Proof of Mader's Conjecture

Theorem 42. *For every $k \geq 3$, $K_{2k+2} - (1 - factor)$ is the only $(2k,k)$-graph.*

Proof. Let f be defined as in Theorem 41.

We start in the following, general setting. Let $n \geq 2$, $k_0 \geq 1$, and $P \subseteq \mathbb{N}$ be fixed for a while. For every $i \geq 0$, let $\mathcal{G}_{n,k_0,P,i}$ be the class of $(n+i, k_0+i)$-graphs of order $|V(G)| = n+i+q$ for some $q \in P$, and define

$$\begin{aligned} n_i &:= n_i(n, k_0, P) \\ &:= \min\{f(G) : G \in \mathcal{G}_{n,k_0,P,i}\}. \end{aligned}$$

(We will vary n, k_0, P later on.) Provided that $n_0 \geq 1$, we are interested in lower bounds for n_i, $i \geq 1$. Define $p := \min(P)$. Let us consider the following recursion, starting with some fixed m_0:

$$m_{i+1} := m_i + \left\lceil \frac{p \cdot m_i}{n+i+1} \right\rceil \text{ for } i \geq 0.$$

We claim that if $n_0 \geq m_0 \geq 1$ then $n_i \geq m_i \geq 1$ for all non-negative integers i. To see this inductively, suppose that $n_i \geq m_i \geq 1$ for some $i \geq 0$. Then every graph in $\mathcal{G}_{n,k_0,P,i}$ satisfies $f(G) \geq m_i$. Let $G \in \mathcal{G}_{n,k_0,P,i+1}$ and take $x \in V(G)$. Since $G - x$ is a graph in $\mathcal{G}_{n,k_0,P,i}$, it follows $f(G - x) \geq m_i$. By Theorem 41, we have

$$\begin{aligned} f(G) &\geq \frac{|V(G)| \cdot m_i}{n+i+1} \\ &\geq m_i + \frac{p \cdot m_i}{n+i+1}. \end{aligned}$$

It follows $n_{i+1} \geq m_{i+1}$, as it has been claimed. Table 1 lists $m_0, \dots, m_i$ for several settings of the parameters m_0, p, n, and i. The first two columns will be used as an index for further references. The last column is always of the form $(2k-1, k-1)$, and the corresponding value in the second last column gives a lower bound m_i to the maximum number of disjoint fragments of cardinality at most 2 in a $(2k-1, k-1)$ graph of order at least $2k-1+8$ (Blocks I. and II.) or of order $2k-1+6$ or $2k-1+7$ (Blocks III. and IV.), respectively. (So the last two columns encode results on $(2k-1, k-1)$-graphs.)

The reader is invited to check these values, by a smart computer program, or by hand. We will come back to these values soon.

Let us have a look at some small cases in advance, by discussing the results encoded in the last two columns of Block II. and Block IV. of Table 1.

Consider the class $\mathcal{G}_{n:=13,k_0:=3,P:=\mathbb{N}_{\geq 8},0}$, where $\mathbb{N}_{\geq 8} := \mathbb{N} - \{1, \dots, 7\}$. From Theorem 36 we know $n_i \geq 2 =: m_i$, and from the first line in Block II. of Table 1 we obtain $n_6 \geq m_6 = 36$, which implies that every $(19,9)$-graph on at least $19+8$ vertices contains 36 disjoint fragments of cardinality at most 2. Considering the classes $\mathcal{G}_{9,2,\mathbb{N}_{\geq 8},0}$, $\mathcal{G}_{8,2,\mathbb{N}_{\geq 8},0}$, and $\mathcal{G}_{7,2,\mathbb{N}_{\geq 8},0}$ together with the second, third, and fourth line of Block II. in Table 1, respectively, and together with Theorem 35, Theorem 35, and Theorem 38, respectively, we derive analogously the results that every $(13,6)$-graph on at least $13+8$ vertices contains 20 disjoint fragments of cardinality at most 2, that every $(11,5)$-graph on at least $11+8$ vertices contains 14 disjoint fragments of cardinality at most 2, and that every $(9,4)$-graph on at least $9+8$ vertices contains 12 disjoint fragments of cardinality at most 2, respectively.

Considering $\mathcal{G}_{13,3,\{6,7\},0}$, $\mathcal{G}_{8,2,\{6,7\},0}$, $\mathcal{G}_{7,2,\{7\},0}$, and $\mathcal{G}_{7,2,\{6\},0}$ together with the first, second, third, and fourth line in Block IV. of Table 1, respectively, and together with Theorem 37, Theorem 39, Theorem 39, and Theorem 40, respectively, we obtain, as above, the results that every $(19,9)$-graph on $19+6$ or $19+7$ vertices contains 25 disjoint fragments of cardinality at most 2, that every $(11,5)$-graph on $11+6$ or $11+7$ vertices contains 13 disjoint fragments of cardinality at most 2, that every $(9,4)$-graph on $9+7$ vertices contains 11 disjoint fragments of cardinality at most 2, and that every $(9,4)$-graph on $9+6$ vertices contains 12 disjoint fragments of cardinality at most 2, respectively.

For an analysis of the behaviour of the sequence (m_i) it is worthwhile to observe that for small p, most of the "contribution" to some m_i has been "gained by rounding fractions to the top". To be more precise, removing the rounding brackets from the recursion formula and consider the m_i as real numbers, in order to simplify the analysis, produces a far, far smaller

Table 1.

	(n,k_0)	p	i	m_0	$m_1,\dots,m_{i-1}$	m_i	$(n+i,k_0+i)$
I.	(29,7)	8	14	2,	3,4,5,7,9,12,15,19,23,28,34,41,49,	59	(43,21)
	(28,7)	8	13	2,	3,4,6,8,10,13,16,20,25,31,38,46,	55	(41,20)
	(26,6)	8	13	2,	3,4,6,8,11,14,18,23,29,36,44,54,	66	(39,19)
	(25,6)	8	12	2,	3,4,6,8,11,14,18,23,29,36,44,	54	(37,18)
	(24,6)	8	11	2,	3,4,6,8,11,14,18,23,29,36,	45	(35,17)
	(23,6)	8	10	2,	3,4,6,8,11,15,19,24,30,	38	(33,16)
	(21,5)	8	10	2,	3,5,7,10,14,19,25,32,41,	52	(31,15)
	(20,5)	8	9	2,	3,5,7,10,14,19,25,33,	43	(29,14)
	(19,5)	8	8	2,	3,5,7,10,14,19,25,	33	(27,13)
	(17,4)	8	8	2,	3,5,7,10,14,19,26,	35	(25,12)
	(16,4)	8	7	2,	3,5,8,12,17,24,	33	(23,11)
	(15,4)	8	6	2,	3,5,8,12,17,	24	(21,10)
	(14,4)	8	5	2,	4,6,9,13,	19	(19,9)
	(12,3)	8	5	2,	4,7,11,17,	25	(17,8)
	(11,3)	8	4	2,	4,7,11,	17	(15,7)
	(10,3)	8	3	2,	4,7,	12	(13,6)
	(9,3)	8	2	2,	4,	7	(11,5)
	(7,2)	8	2	2,	4,	8	(9,4)
II.	(13,3)	8	6	2,	4,7,11,17,25,	36	(19,9)
	(9,2)	8	4	2,	4,7,12,	20	(13,6)
	(8,2)	8	3	2,	4,8,	14	(11,5)
	(7,2)	8	2	3,	6,	12	(9,4)
III.	(27,5)	6	16	2,	3,4,5,6,8,10,12,15,18,21,25,29,34,39,45,	52	(43,21)
	(26,5)	6	15	2,	3,4,5,6,8,10,12,15,18,21,25,29,34,40,	46	(41,20)
	(25,5)	6	14	2,	3,4,5,7,9,11,14,17,20,24,28,33,39,	45	(39,19)
	(23,4)	6	14	2,	3,4,5,7,9,11,14,17,21,25,30,36,42,	49	(37,18)
	(22,4)	6	13	2,	3,4,5,7,9,11,14,17,21,25,30,36,	43	(35,17)
	(21,4)	6	12	2,	3,4,5,7,9,11,14,17,21,26,31,	37	(33,16)
	(20,4)	6	11	2,	3,4,6,8,10,13,16,20,25,30,	36	(31,15)
	(19,4)	6	10	2,	3,4,6,8,10,13,16,20,25,	31	(29,14)
	(17,3)	6	10	2,	3,4,6,8,11,14,18,23,29,	36	(27,13)
	(16,3)	6	9	2,	3,4,6,8,11,14,18,23,	29	(25,12)
	(15,3)	6	8	2,	3,5,7,10,13,17,22,	28	(23,11)
	(14,3)	6	7	2,	3,5,7,10,14,19,	25	(21,10)
	(13,3)	6	6	2,	3,5,7,10,14,	19	(19,9)
	(11,2)	6	6	2,	3,5,8,12,17,	23	(17,8)
	(10,2)	6	5	2,	4,6,9,13,	19	(15,7)
	(9,2)	6	4	2,	4,7,11,	17	(13,6)
	(8,2)	6	3	2,	4,7,	11	(11,5)
	(7,2)	6	2	2,	4,	7	(9,4)
IV.	(13,3)	6	6	3,	5,7,10,14,19,	25	(19,9)
	(8,2)	6	3	3,	5,8,	13	(11,5)
	(7,2)	7	2	3,	6,	11	(9,4)
	(7,2)	6	2	4,	7,	12	(9,4)

bound, which even turns out to be too bad for our purposes. Therefore, we have too find another representation of m_i, which is more tailored to our application.

Clearly, $(m_i)_{i=0,1,2,\dots}$ is a monotonely increasing sequence. Indeed, the same is true for $(m_{i+1}-m_i)_{i=0,1,2,\dots}$; to prove this it is sufficient to show that $\frac{p\cdot m_{i+1}}{n+(i+1)+1} \geq \frac{p\cdot m_i}{n+i+1}$ holds for all $i \geq 0$, which is equivalent to $p\cdot m_{i+1} \geq p\cdot m_i + \frac{p\cdot m_i}{n+i+1}$. Using the recursion equation for m_{i+1}, this is equivalent to $p\cdot m_i + p\cdot \lceil \frac{p\cdot m_i}{n+i+1}\rceil \geq p\cdot m_i + \frac{p\cdot m_i}{n+i+1}$, which is obviously true. Hence $(m_{i+1} - m_i)_{i=0,1,2,\dots}$ is monotonely increasing.

Let $m_0 \geq 1$ be arbitrary and set

$$d_0 := 0 \text{ and, recursively for } \ell \geq 1,$$

$$d_\ell := \left\lfloor \frac{\ell\cdot(n+1) + p\cdot(\sum_{j=0}^{\ell-1} d_j - m_0)}{\ell\cdot(p-1)} + 1 \right\rfloor .$$

We claim that if $n \geq (m_0-1)\cdot p$ then for all $\ell \geq 0$ the following is true.

(1) If $\ell \geq 1$ and $d_{\ell-1} \leq i \leq d_\ell$ then $m_i = (m_0 - \sum_{j=0}^{\ell-1} d_j) + \ell\cdot i.$

(2) If $\ell \geq 0$ and $i \geq 0$ then $m_{i+1} - m_i > \ell \longleftrightarrow i \geq d_\ell.$

We first show that $(d_\ell)_{\ell=0,1,2,\dots}$ is a monotonely increasing sequence, by proving $d_{\ell+1} \geq d_\ell$ for all $\ell \geq 0$ inductively on ℓ. The induction starts since $d_1 = \lfloor \frac{n+1-p\cdot m_0}{p-1} + 1\rfloor = \lfloor \frac{n-(p-1)-p\cdot(m_0-1)}{p-1} + 1\rfloor = \lfloor \frac{n-(m_0-1)\cdot p}{p-1}\rfloor \geq 0 = d_0$. For the induction step, let $\ell \geq 1$ and suppose $d_\ell \geq d_{\ell-1} \geq \cdots \geq d_1 \geq d_0$. To prove $d_{\ell+1} \geq d_\ell$ it is sufficient to prove that $\frac{(\ell+1)\cdot(n+1)+p\cdot(\sum_{j=0}^{\ell} d_j - m_0)}{\ell+1} \geq \frac{\ell\cdot(n+1)+p\cdot(\sum_{j=0}^{\ell-1} d_j - m_0)}{\ell}$, which is equivalent to $\ell\cdot(\ell+1)\cdot(n+1) + \ell\cdot p\cdot(\sum_{j=0}^{\ell} d_j - m_0) \geq (\ell+1)\cdot\ell\cdot(n+1) + (\ell+1)\cdot p\cdot(\sum_{j=0}^{\ell-1} d_j - m_0)$. The latter is equivalent to $\ell\cdot(\sum_{j=0}^{\ell-1} d_j - m_0) + \ell\cdot d_\ell \geq \ell\cdot(\sum_{j=0}^{\ell-1} d_j - m_0) + (\sum_{j=0}^{\ell-1} d_j - m_0)$, and to $\ell\cdot d_\ell \geq \sum_{j=0}^{\ell-1} d_j - m_0$, and this is true since, by hypothesis, each of the ℓ summands of $\sum_{j=0}^{\ell-1} d_j$ is less than or equal to d_ℓ. This proves that $(d_\ell)_{\ell=0,1,2,\dots}$ is a monotonely increasing sequence. In particular, $d_\ell \geq 0$ for all $\ell \geq 0$.

Before we prove (1) and (2) let's observe the following:

$$(3)\quad \begin{array}{llcl} \multicolumn{4}{l}{\text{If } \ell \geq 0 \text{ and } i \geq 0 \text{ and } m_i = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot i} \\ \multicolumn{4}{l}{\text{then the following statements are equivalent:}} \\ & m_{i+1} - m_i & \leq & \ell+1 \\ \longleftrightarrow & \lceil \frac{p\cdot m_i}{n+i+1} \rceil & \leq & \ell+1 \\ \longleftrightarrow & \frac{p\cdot m_i}{n+i+1} & \leq & \ell+1 \\ \longleftrightarrow & p\cdot m_i & \leq & (\ell+1)\cdot(n+i+1) \\ \longleftrightarrow & & & p\cdot((m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot i) \\ & & \leq & (\ell+1)\cdot(n+i+1) \\ \longleftrightarrow & & & (p-1)\cdot(\ell+1)\cdot i \\ & & \leq & (n+1)\cdot(\ell+1) + p\cdot(\sum_{j=0}^{\ell} d_j - m_0) \\ \longleftrightarrow & i+1 & \leq & \frac{(n+1)\cdot(\ell+1)+p\cdot(\sum_{j=0}^{\ell} d_j - m_0)}{(p-1)\cdot(\ell+1)} + 1 \\ \longleftrightarrow & i+1 & \leq & \left\lfloor \frac{(n+1)\cdot(\ell+1)+p\cdot(\sum_{j=0}^{\ell} d_j - m_0)}{(p-1)\cdot(\ell+1)} + 1 \right\rfloor \\ \longleftrightarrow & i+1 & \leq & d_{\ell+1}. \end{array}$$

Now we prove (1) and (2), by induction on ℓ. They are both obvious for $\ell = 0$. Suppose now that they are true for some fixed $\ell \geq 0$. We show that they are true for $\ell+1$ as well. Since for $i \geq d_\ell \geq 0$ we have $m_{i+1} - m_i \geq \ell+1$ by hypothesis (2), we obtain by (3):

$$(4)\quad \begin{array}{l} \text{If } i \geq d_\ell \text{ and } m_i = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot i \\ \text{then} \\ m_{i+1} - m_i = \ell+1 \longleftrightarrow i+1 \leq d_{\ell+1}. \end{array}$$

Now we prove (1) for $\ell+1$ instead of ℓ by induction on i, starting with $i = d_\ell$. For $\ell+1 = 1$ and $i = d_\ell = d_0 = 0$ it is true, since $m_{d_\ell} = m_0 = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot d_\ell$. If $\ell \geq 1$ then the induction on i starts with $i = d_\ell$, too, since we may apply hypothesis (1) for ℓ to $i = d_\ell$ (as we have seen above, $i = d_\ell \geq d_{\ell-1}$): We obtain $m_{d_\ell} = (m_0 - \sum_{j=0}^{\ell-1} d_j) + \ell\cdot d_\ell = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot d_\ell$. For the induction step, suppose that (1) holds for $\ell+1$ instead of ℓ and some $i \geq d_\ell$, $i < d_{\ell+1}$. We have to show that (1) holds for $\ell+1$ instead of ℓ and for $i+1$ instead of i. From (4) it follows $m_{i+1} - m_i = \ell+1$, and, thus, $m_{i+1} = m_i + (\ell+1) = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot(i+1)$, which finally proves (1) for $\ell+1$ instead of ℓ.

To accomplish the induction on ℓ, it remains to show that (2) holds for $\ell+1$ instead of ℓ. Since $d_{\ell+1} \geq d_\ell$, we may apply (1) for $\ell+1$ instead of ℓ to $i = d_{\ell+1}$ and obtain $m_{d_{\ell+1}} = (m_0 - \sum_{j=0}^{\ell} d_j) + (\ell+1)\cdot d_{\ell+1}$. Since $d_{\ell+1}+1 \not\leq d_{\ell+1}$, it follows from (3), applied to $i = d_{\ell+1}$, that $m_{d_{\ell+1}+1} - m_{d_{\ell+1}} > \ell+1$. Since $(m_{i+1} - m_i)_{i=0,1,2,\ldots}$ is a monotonely increasing sequence, $m_{i+1} - m_i > \ell+1$ follows *for all* $i \geq d_{\ell+1}$.

Table 2.

m_0	p	d_0^*	d_1^*	d_2^*	d_3^*	d_4^*
2	8	0	$\frac{1}{7}\cdot n-\frac{8}{7}$	$\frac{11}{49}\cdot n-\frac{32}{49}$	$\frac{97}{343}\cdot n-\frac{104}{343}$	$\frac{789}{2401}\cdot n-\frac{68}{2401}$
2	6	0	$\frac{1}{5}\cdot n-\frac{6}{5}$	$\frac{8}{25}\cdot n-\frac{18}{25}$	$\frac{51}{125}\cdot n-\frac{46}{125}$	$\frac{299}{625}\cdot n-\frac{54}{625}$

What happens for $i<d_{\ell+1}$? If $i<d_\ell$ then $m_{i+1}-m_i\leq\ell$ by hypothesis (2). If $i\geq d_\ell$ then we may apply (1) for ℓ+1 instead of ℓ to i and to i+1 instead of i. It follows $m_i=(m_0-\sum_{j=0}^{\ell}d_j)+(\ell+1)\cdot i$ and $m_{i+1}=(m_0-\sum_{j=0}^{\ell}d_j)+(\ell+1)\cdot(i+1)$, which yields $m_{i+1}-m_i=\ell+1$. Therefore, $m_{i+1}-m_i\leq\ell+1$ for all $i<d_{\ell+1}$, which finally proves (2) for $\ell+1$ instead of ℓ.

Hence we proved (1) and (2). This leads to

(5) If $n\geq(m_0-1)\cdot p$, $\ell\geq 0$, and $i\geq d_\ell$ then $m_i\geq(m_0-\sum_{j=0}^{\ell}d_j)+(\ell+1)\cdot i$.

To prove this, keep ℓ being fixed and perform induction on i. The induction starts for $i=d_\ell$ by (1) applied to $\ell+1$ instead of ℓ, and the induction step works, since, by (2), for $i\geq d_\ell$, $m_{i+1}=m_i+(m_{i+1}-m_i)\geq m_i+(\ell+1)$. Using the induction hypothesis, (5) follows.

For the further investigation, it is more convenient to look at "unrounded" d_ℓ. Set

$$d_0^*:=0 \text{ and, recursively for } \ell\geq 1,$$
$$d_\ell^*:=\frac{\ell\cdot(n+1)+p\cdot(\sum_{j=0}^{\ell-1}d_j^*-m_0)}{\ell\cdot(p-1)}+1,$$

for the same fixed parameters n,m_0,p. Two particular parameter settings for m_0 and p will be of relevance later on. Table 2 shows the values of $d_0^*,\dots,d_4^*$ for them.

We obtain inductively that $d_\ell^*\geq d_\ell$ for all $\ell\geq 0$. Hence, by (5) we obtain

(6) If $n\geq(m_0-1)\cdot p$, $\ell\geq 0$, and $i\geq d_\ell$ (in particular, if $i\geq d_\ell^*$) then

$$\begin{array}{rcl} n_i & \geq & m_i \\ & \geq & (m_0-\sum_{j=0}^{\ell}d_j)+(\ell+1)\cdot i \\ & \geq & (m_0-\sum_{j=0}^{\ell}d_j^*)+(\ell+1)\cdot i \\ & =: & q_{\ell,i}. \end{array}$$

Table 3 lists $q_{4,i}$ for the two parameter settings as above. Again, the reader is invited to verify them.

Now we are prepared to prove the theorem. Take a $(2k,k)$-graph G non-isomorphic to $K_{2k+2}-(1-factor)$. By Theorem 24, we may assume that $k\geq 5$. By Theorem 28, we may assume $|V(G)|\geq 2k+6$.

Table 3.

m_0	p	$q_{4,i}$
2	8	$-\frac{2350}{2401}\cdot n+\frac{9910}{2401}+5\cdot i$
2	6	$-\frac{879}{625}\cdot n+\frac{2734}{625}+5\cdot i$

Case 1. Let's assume first that $|V(G)|\geq 2k+8$.

Let $j:=\lfloor\frac{14k-11}{19}\rfloor$ be the largest integer such that $2k-j<\frac{24}{5}(k-j)-2$ holds. Set $n:=2k-j$ and consider for $i\geq 0$ the classes $\mathcal{G}_i:=\mathcal{G}_{n,k_0:=k-j,P:=\mathbb{N}_{\geq 8},i}$. Let $p=min(P)=8$ and $m_0=2$. Since $n<\frac{24}{5}(k-j)-2$, every graph in $\mathcal{G}_0$ has two fragments of cardinality 1 or 2 each by Theorem 33. This implies $n_0\geq 2=:m_0$.

Claim 1. $n_{j-1}>2k$ for all $k\geq 5$.

We would like to apply the bound of (6), and so we have to take care that $n\geq(m_0-1)\cdot p=8$ and that $j-1\geq d_4^*$ hold. As one easily checks, the first inequality is true for all $k\geq 7$ (even for $k=6$). From Table 2 we obtain $d_4^*=(789/2401)\cdot n-68/2401=(789/2401)\cdot(2k-j)-68/2401$, and thus $j-1\geq d_4^*$ is equivalent to $3190\cdot j\geq 1578\cdot k+2333$. Since $j\geq(14k-11)/19-18/19=(14/19)\cdot k-29/19$, it suffices to consider the inequality $3190\cdot(14/19)\cdot k-3190\cdot(29/19)\geq 1578\cdot k+2333$, which is equivalent to $14678\cdot k\geq 136837$ and so turns out to be true for all $k\geq 10$. We proceed by investigating for which $k\geq 10$ we have $q_{4,j-1}>2k$. From Table 3 we obtain $q_{4,j-1}=(-2350/2401)\cdot n+9910/2401+5\cdot(j-1)=(-2350/2401)\cdot(2k-j)+9910/2401+5\cdot(j-1)$, and thus $q_{4,j-1}>2k$ is equivalent to $14355\cdot j>9502\cdot k+2095$. Again, it suffices to consider the inequality $14355\cdot(14/19)\cdot k-14355\cdot(29/19)>9502\cdot k+2095$, which is equivalent to $20432\cdot k>456100$ and so turns out to be true for all $k\geq 23$. Hence we proved Claim 1 for all $k\geq 23$.

What about the values $k\in\{5,\dots,22\}$? Block I. of Table 1 shows the values $m_0,\dots,m_{j-1}$ for each of them (where $i=j-1$, hence in the rightmost column one finds the pair $(n+i,k_0+i)=(2k-1,k-1)$). As one can see, $m_i=m_{j-1}>2k$ holds for all $k\in\{5,\dots,22\}-\{5,6,7,10\}$.

Since $\mathcal{G}_{14,4,\mathbb{N}_{\geq 8},5}=\mathcal{G}_{13,3,\mathbb{N}_{\geq 8},6}$, $\mathcal{G}_{10,3,\mathbb{N}_{\geq 8},3}=\mathcal{G}_{9,2,\mathbb{N}_{\geq 8},4}$, and $\mathcal{G}_{9,3,\mathbb{N}_{\geq 8},2}=\mathcal{G}_{8,2,\mathbb{N}_{\geq 8},3}$, it follows from the observations in the paragraph following Table 1 that $n_{j-1}>2k$ for $k\in\{5,6,7,10\}$ as well.

This proves Claim 1.

By definition, the class $\mathcal{G}_{j-1}$ is the class of $(2k-1,k-1)$-graphs on at least $2k-1+8$ vertices. From Theorem 23 we know that G contains a vertex x of degree $2k$. Since $G-x$ is a $(2k-1,k-1)$-graph on at least $2k-1+8$ vertices, we know from Claim 1, that an end cover of $G-x$ has cardinality at least

$2k+1$. On the other hand, $N(x)$ is an end cover set of $G-x$ of cardinality $2k$ – a contradiction.

Case 2. Now let's assume that $|V(G)| \in \{2k+6, 2k+7\}$ and $k \geq 5$.

Let $j := \lfloor \frac{4k-1}{5} \rfloor$ be the largest integer such that $2k-j < 6 \cdot (k-j)$. Reset $n := 2k-j$ and consider $\mathcal{G}_i := \mathcal{G}_{n,k_0:=k-j,P:=\{6,7\},i}$. Since $n < 6k$, every graph in $\mathcal{G}_0$ must contain two fragments of cardinality 1 or 2 by Theorem 32. Hence we obtain $n_0 \geq 2 =: m_0$. Let $p = \min(P) = 6$.

Claim 2. $n_{j-1} > 2k$ for all $k \geq 5$.

We would like to apply the bound of (6), and so we have to take care that $n \geq (m_0-1) \cdot p = 6$ and $j-1 \geq d_4^*$ hold. As one easily checks, the first inequality is true for all $k \geq 5$. From Table 2 we obtain $d_4^* = (299/625) \cdot n - 54/625 = (299/625) \cdot (2k-j) - 54/625$, and thus $j-1 \geq d_4^*$ is equivalent to $924 \cdot j \geq 598 \cdot k + 571$. Since $j \geq (4k-1)/5 - 4/5 = (4/5) \cdot k - 1$, it suffices to consider the inequality $924 \cdot (4/5) \cdot k - 924 \geq 598 \cdot k + 571$, which is equivalent to $706 \cdot k \geq 7475$ and so turns out to be true for all $k \geq 11$. We proceed by investigating for which $k \geq 11$ we have $q_{4,j-1} > 2k$. From Table 3 we obtain $q_{4,j-1} = (-879/625) \cdot n - 254/125 + 5 \cdot (j-1) = (-879/625) \cdot (2k-j) - 254/125 + 5 \cdot (j-1)$, and thus $q_{4,j-1} > 2k$ is equivalent to $4004 \cdot j > 3008 \cdot k + 391$. Again, it suffices to consider the inequality $4004 \cdot (4/5) \cdot k - 4004 > 3008 \cdot k + 391$, which is equivalent to $976 \cdot k > 21975$ and so turns out to be true for all $k \geq 44$. Hence we proved Claim 1 for all $k \geq 23$. The exact values for m_{j-1} and $k \in \{5, \ldots, 22\}$ can be found in Block III. of Table 1. For all $k \notin \{5, 6, 10\}$ they turn out to be greater than $2k$.

Since $\mathcal{G}_{7,2,\{6,7\},2} = \mathcal{G}_{7,2,\{6\},2} \cup \mathcal{G}_{7,2,\{7\},2}$ we obtain from the observations in the second paragraph following Table 1 that $m_{j-1} > 2k$ holds for $k \in \{5, 6, 7\}$, too (again, $i = j-1$, and so the rightmost column contains the pairs $(n+i, k_0+i) = (2k-1, k-1)$).

This finally proves Claim 2.

By definition, $\mathcal{G}_{j-1}$ is the class of $(2k-1, k-1)$-graphs on $2k+5$ or $2k+6$ vertices. From Theorem 23 we know that G contains a vertex x of degree $2k$. Since $G-x$ is a $(2k-1, k-1)$-graph on $2k-1+6$ or $2k-1+7$ vertices, we know from Claim 2, that an end cover of $G-x$ has cardinality at least $2k+1$. On the other hand, $N(x)$ is an end cover of $G-x$ of cardinality $2k$ – a contradiction. ∎

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